Proof of a Null Penrose Conjecture Using a New Quasi-local Mass

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
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Abstract

In the theory of general relativity, the Penrose conjecture claims a lower bound for the mass of a spacetime in terms of the area of an outermost horizon, if one exists. In physical terms, this conjecture is a geometric formulation of the statement that the total mass of a spacetime is at least the mass of any black holes that are present, assuming non-negative energy density. For the geometry of light-rays emanating off of a black hole horizon (called a nullcone), the Penrose conjecture can be reformulated to the so-called Null Penrose Conjecture (NPC). In this thesis, we define an explicit quasi-local mass functional that is non-decreasing along all foliations (satisfying a convexity assumption) of nullcones. We use this new functional to prove the NPC under fairly generic conditions.

Dedicated to God, my wife, and my parents.

Contents

List of Figures

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1

Introduction

In the framework of Albert Einstein's theory of General Relativity, the presence of matter induces curvature of the four dimensional fabric we inhabit called *space-time*. Our perception of gravity is a direct consequence of this curvature and the reaction of entities to its presence. Mathematically, we recognize a spacetime as a semi-Riemannian manifold $(\mathcal{M}^4, g)$, where $\mathcal{M}$ is a four dimensional manifold with metric $g(\cdot,\cdot)$ of signature $(-,+,+,+)$. Throughout this thesis, we will be setting the universal gravitational constant and the speed of light to unity. The fundamental bridge between our geometric and physical interpretations of the framework is given by the Einstein equation,

$$G = 8\pi T$$

where T represents the physical *stress-energy* tensor, and $G = Ric_g - \frac{1}{2}R_g g$ is the *Einstein tensor*. From a physical perspective, the tensor $T(\cdot,\cdot)$ signals the presence of matter in space-time by the measurement of energy-momentum at any given point. With regards to geometry, since the Einstein tensor G is constructed from the Ricci curvature tensor $Ric_g(\cdot,\cdot)$ and scalar curvature R_g (of the metric g), it measures geometric curvature. The Einstein equation therefore conflates both these interpretive lenses providing a theory whereby matter curves spacetime.

1.1 The Penrose conjecture

In 1973, Sir Roger Penrose ([23]) conjectured that the mass contributed by a collection of black holes should be no less than $\sqrt{\frac{|\Sigma|}{16\pi}}$, where $|\Sigma|$ is the total combined area of all black hole horizons Σ in our spacetime. Alternatively,

$$M \geqslant \sqrt{\frac{|\Sigma|}{16\pi}}, \tag{1.1}$$

where M is the total mass. One of the fundamental ingredients of Penrose's heuristic argument was the use of *cosmic censorship*. As a statement on the global future evolution of a system, cosmic censorship is essential for an existence theorem in general relativity. This hypothesis asserts that generically all spacetime singularities are hidden from the rest of the universe by black holes. Since singularities are shown to exist for physically reasonable spacetimes (by the famous work of Hawking and Penrose [13]), these semi-permeable information barriers serve to prevent their chaotic physical implications from influencing our deterministic system. Finding a counterexample to (1.1) would likely indicate a failure of cosmic censorship (in fact this was Penrose's original motivation for studying (1.1)), and, conversely, a proof of (1.1) would provide indirect support to its validity. Another important ingredient in Penrose's argument was to assume the *Dominant Energy Condition*. This condition, via the energy-momentum tensor T, imposes local curvature constraints to model a spacetime with non-negative energy density. From this perspective, (1.1) in essence claims that this non-negative energy density must aggregate (in analogy to our classical understanding of mass) to at least the black hole contributions. This would also refine the famous *positive mass theorem* of Schoen and Yau ([26],[27]) discussed in Section 1.3.

Recognizing the implications of the Penrose conjecture on the immensely successful framework laid out by Einstein, mathematicians and physicists have spent considerable effort in the last forty years towards a rigorous proof. Thus far, attempts have centered on two main approaches. The traditional approach centers on analyzing initial data $(\mathcal{N}^3, \bar{g}, K)$ where $(\mathcal{N}, \bar{g})$ denotes a Riemannian slice of the spacetime $\mathcal{M}$ of extrinsic curvature (or second fundamental form) K. We briefly discuss this setting and the beautiful results that answer special cases in Section 1.3. The study of this thesis however, involves an approach involving null slices of $\mathcal{M}$. A *null hypersurface* $\Omega \hookrightarrow \mathcal{M}$ is represented by a three-dimensional manifold Ω on which the induced metric $\gamma = g|_{\Omega}$ is degenerate. A major appeal of this setting is the existence of a null tangent vector $\underline{L} \in \Gamma(T\Omega)$ generating null geodesics that rule Ω. For this very reason, many approaches in the Riemannian setting involving difficult geometric partial differential equations reduce in the null setting to an analysis of ordinary differential equations. To enrich the discussion of these two approaches and to help build an intuition for the fundamental ideas that underpin this thesis we spend some time in the setting of Einstein's theory of *Special Relativity.*

1.2 Minkowski spacetime

In a vacuum, namely $G = 8\pi T = 0$, our simplest model is given by the flat Minkowski space $\mathbb{R}^4_1 := (\mathbb{R}^4, g)$ whereby g is given by the quadratic form

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

in a global chart (t, x, y, z). Although highly restrictive, Minkowski spacetime originally served as the springboard for the development of Albert Einstein's general theory as well as an initial gateway to some of the most fundamental discoveries in twentieth century physics. In this section, we invest some time introducing key aspects of the Minkowski spacetime that will serve as a backdrop throughout this thesis.

1.2.1 *Particles and Observers*

An immediate consequence of the vector space structure is the canonical isomorphism between the spacetime and its tangent spaces, whereby $\mathbb{R}^4_1 \cong T_{\vec{q}}\mathbb{R}^4_1$ for any point $\vec{q} \in \mathbb{R}^4_1$. This warrants our ability to unambiguously pull-back the quadratic form associated to $g = \langle \cdot, \cdot \rangle$. Therefore, given a non-trivial vector $v \in \mathbb{R}^4_1$, we immediately notice its existence within one of three categories. If $\langle v, v\rangle < 0$ we say v is *timelike*, if $\langle v, v\rangle > 0$ we say v is *spacelike*, and if $\langle v, v\rangle = 0$ we say v is *null*. We define the length of $v \in \mathbb{R}^4_1$ with $|v| := \sqrt{|\langle v, v\rangle|}$. It will also be useful to distinguish spacelike vectors from non-spacelike vectors by referring to the latter as *causal* vectors. Taking the globally defined timelike vector ∂_t to define a pointwise 'direction of time', any causal vector v can then be identified as either *future pointing* if $\langle v, \partial_t\rangle < 0$ or *past pointing* if $\langle v, \partial_t\rangle > 0$. A material particle is then represented in $\mathbb{R}^4_1$ by a curve $\alpha(\tau) : I \to \mathbb{R}^4_1$ which has a unit future-pointing timelike velocity α' (i.e. such that $\langle \alpha', \alpha'\rangle = -1$). Given the frame $\{\partial_t, \partial_x, \partial_y, \partial_z\} = \{\partial_t, \partial_i\}$ the energy-momentum for a particle of mass M is given by

$$\mathcal{T} := M\alpha' = E\partial_t + P^i\partial_i.$$

For convenience we may denote $\mathcal{T} = (E, \vec{P})$, where E is the measured energy of the particle and $\vec{P}$ its linear momentum. We see that $M^2 = -\langle \mathcal{T}, \mathcal{T}\rangle = E^2 - |\vec{P}|^2$, which is independent of our choice of frame (i.e. under an arbitrary isometry $\phi : \mathbb{R}^4_1 \to \mathbb{R}^4_1$), whereas E and $\vec{P}$ are not.

In General Relativity, 'free-falling' particles are characterized by the restriction that α be geodesic i.e. $\alpha'' = 0$. In $\mathbb{R}^4_1$ we conclude therefore that free falling particles have constant energy-momentum.

Within the Lie group of isometries of $\mathbb{R}^4_1$, we will concentrate on the subgroup fixing the origin called the Lorentz group. In particular, the connected component of the identity (or the restricted Lorentz group), is generated by pure rotations of space ($SO(3)$) and the *Lorentz boosts* $\phi_{\vec{v}} : \mathbb{R}^4_1 \to \mathbb{R}^4_1$:

$$(E, \vec{p}) \xrightarrow{\phi_{\vec{v}}} \Big(\mu_{\vec{v}}(E - \vec{v}\cdot\vec{p}), \vec{p} + (\mu_{\vec{v}}E + \frac{(\gamma-1)(\vec{p}\cdot\vec{v})}{|\vec{v}|^2})\vec{v}\Big) \tag{1.2}$$

for some $\vec{v} \in \mathring{B}^3 \subset \mathbb{R}^3$ and $\mu_{\vec{v}}^{-1} := \sqrt{1 - |\vec{v}|^2}$. It's easily seen that all future directed unit timelike vectors are uniquely reached by a boost of ∂_t. As a result, given any free falling future directed trajectory α we are able to boost to a frame where it remains *at rest*, i.e. $\alpha^i = 0$. Equivalently, we may identify these trajectories as co-observers within Minkowski spacetime each with an associated boosted frame of reference. A free falling particle is therefore viewed from an observer at rest (relative to the motion of the particle) to have energy-momentum

$$\mathcal{T} = (M, \vec{0})$$

giving Einstein's famous identity that $E = M$ (or $E = Mc^2$ when the speed of light, c, is not set to unity).

1.2.2 Nullcones

Given a linear map $\phi : \mathbb{R}^4_1 \to \mathbb{R}^4_1$ it's an easy exercise to show (forgiving the abuse of notation) that $\phi = d\phi$. Since the Lorentz group is induced by linear maps we therefore observe for any $w \in \mathbb{R}^4_1 \cong T_{\phi_{\vec{v}}(w)}\mathbb{R}^4_1$ that $\langle d\phi_{\vec{v}}(w), d\phi_{\vec{v}}(w)\rangle = \langle w, w\rangle$. As a result, the hyperquadrics of $\mathbb{R}^4_1$

$$H_C := \{v \in \mathbb{R}^4_1 | \langle v, v\rangle = C\}$$

are fixed under the action of $\phi_{\vec{v}}$, which therefore restricts to a diffeomorphism of submanifolds. Whenever $C \neq 0$, $\phi_{\vec{v}}$ furthermore restricts to an isometry of the induced semi-Riemannian submanifold H_C.

Considering the hyperquadric given by $C = 0$, we are finally led to one of the fundamental entities in this thesis and the notion of light in General Relativity. A ray of light (similarly to a free falling particle) is identified by a geodesic trajectory. While a particle is restricted to having a timelike velocity, a light ray $\beta(\lambda)$ instead must satisfy $\langle \beta', \beta' \rangle = 0$. As a result, for a ray of light emanating from the origin in $\mathbb{R}^4_1$ we are handed some $\beta(\lambda) = \lambda v$ whereby $\langle v, v \rangle = 0$. The collection of all past-pointing null geodesics form a null hypersurface $\Omega \hookrightarrow \mathbb{R}^4_1$

$$\Omega := \{v \in \mathbb{R}^4_1 \,|\, \langle v, v \rangle = 0 < \langle v, \partial_t \rangle\}$$

called the *past nullcone* of o. Interestingly, given the characterization of particles and light, we notice that the set $\Omega + \alpha(\tau)$ bounds the causal history of an observer at $\alpha(\tau)$ since no matter or light to the future of this boundary can reach the event $\alpha(\tau)$. Moreover, by time symmetry, no particle released to the future of $\alpha(\tau)$ will be able to escape the future nullcone. The reader may recognize this as a geometric realization of the fact that matter cannot travel faster than the speed of light (which is fixed at $c = 1$).

We conclude this section by showing a one-to-one correspondence between round foliations of Ω and the boosts $\phi_{\vec{v}}$. Since any slicing of $\mathbb{R}^4_1$ by parallel spacelike hyperplanes is uniquely identifiable by their common timelike future normal (and therefore to a boost), they are given as level sets of a boosted 'time function' $\bar{t}$. We start therefore with the trivial slicing associated to our coordinate function t and observe that it induces a foliation of Ω by round spheres (any two of which are homothetic under a radius re-scaling). Since the diffeomorphism $\phi_{\vec{v}}|_\Omega$ is induced by an isometry that isometrically 'tilts' the trivial slicing, intersection with Ω leads to an isometric tilting of the initial foliation. In fact, in spherical polar coordinates we see that the image under $\phi_{\vec{v}}$ of the unit sphere is given by $\Sigma := \{t = r = \omega\}$ for some function $\omega \in \mathcal{F}(\mathbb{S}^2)$ only if the inverse map gives $\phi_{-\vec{v}}(-\omega(\vartheta, \varphi), \omega(\vartheta, \varphi), \vartheta, \varphi) = (-1, 1, f(\vartheta, \varphi), g(\vartheta, \varphi))$.

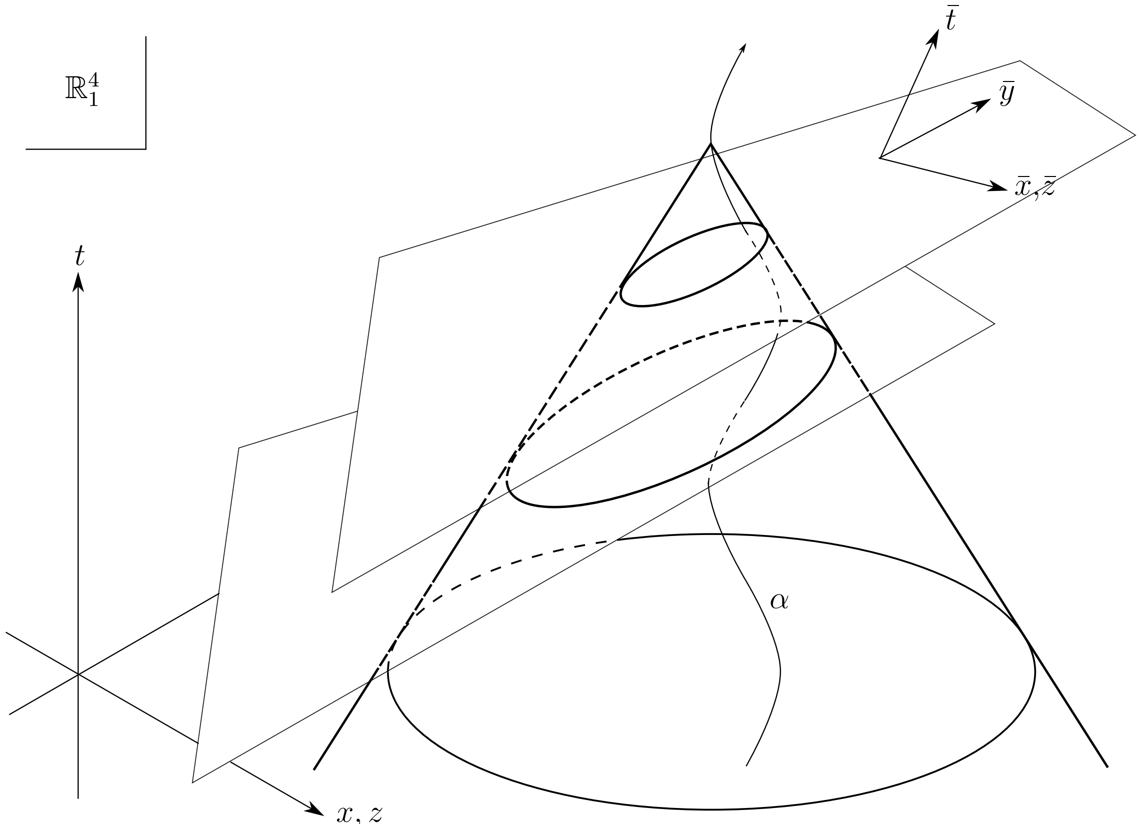

FIGURE 1.1: Particle in $\mathbb{R}^4_1$

From this, (1.2) gives

$$\omega(\vartheta, \varphi) = \frac{1}{\mu_{\vec{v}}(1 - \vec{v} \cdot \vec{n}(\vartheta, \varphi))} \tag{1.3}$$

for $\vec{n}(\vartheta, \varphi)$ the unit position vector in $\mathbb{R}^3$. From the quadratic form in spherical polar coordinates

$$ds^2 = -dt^2 + dr^2 + r^2(d\vartheta^2 + (\sin\vartheta)^2 d\varphi^2),$$

we conclude that Σ has induced round metric $\psi^2 \mathring{\gamma}$ (i.e. Gaussian curvature $\mathcal{K}_{\psi^2\mathring{\gamma}} = 1$) where $\mathring{\gamma}$ is the standard round metric of $\mathbb{S}^2$. Equivalently, for $\mathring{\Delta}$ the Laplacian associated to $\mathring{\gamma}$,

$$1 = \frac{1}{\omega^2}(1 - \mathring{\Delta} \log \omega). \tag{1.4}$$

The converse, namely that any function solving (1.4) must be of the form (1.3) is known ([18]) and follows from the following geometric result (the reader may wish to skip the proof, which we include for completeness):

Lemma 1.2.1. *$\Sigma \hookrightarrow \Omega$ is a round sphere only if Σ is the intersection of Ω with a spacelike hyperplane.*

Proof. Utilizing the spherical polar form of the metric of Minkowski space we conclude as before that Σ must have induced metric $\gamma = \omega^2 \mathring{\gamma}$ for some function $\omega \in \mathcal{F}(\mathbb{S}^2)$. The assumption that Σ be round also gives us that $\mathcal{K}_{\omega^2\mathring{\gamma}} = 1$ (and that ω solves (1.3)). Our result therefore follows as soon as we show the existence of a constant unit timelike vector $N \in \Gamma(T^{\perp}\Sigma)$ for Σ. In the Minkowski spacetime we have the position vector field $P := t\partial_t + x^i\partial_i$ such that $D_X P = X$ for any $X \in \Gamma(T\mathbb{R}^4_1)$, where D is the Levi-Civita connection. From the associated second fundamental form and mean curvature of Σ

$$\mathrm{II}(V, W) = D^{\perp}_V W$$

$$\vec{H} = \mathrm{tr}_{\Sigma}\, \mathrm{II}$$

for $V, W \in \Gamma(T\Sigma)$, we conclude from metric compatibility that $\langle \mathrm{II}, P\rangle|_{\Sigma} = -\gamma$ implies $\langle \vec{H}, P\rangle = -2$. From construction of Ω it also follows that $\langle P, P\rangle|_{\Omega} = 0$ so that $\langle V, P\rangle = \frac{1}{2}V\langle P, P\rangle = 0$, for any $V \in \Gamma(T\Sigma)$. We conclude that Σ has trivial normal bundle with basis $\{P, \vec{H}\} \subset T^{\perp}\Sigma$. Since $\mathrm{II} = aP + b\vec{H}$ for two symmetric 2-tensors a, b we have $-2b = \langle \mathrm{II}, P\rangle = -\gamma$. So for the traceless part of II it follows that $\hat{\mathrm{II}} = \hat{a}P$. From the Gauss equation ([21], pg.100) for $\Sigma \hookrightarrow \mathbb{R}^4_1$ we see

$$\mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H}\rangle + \frac{1}{2}\langle \hat{\mathrm{II}}, \hat{\mathrm{II}}\rangle = 0,$$

so we immediately conclude that $\langle \vec{H}, \vec{H}\rangle = 4$. We now show that $N = P + \frac{1}{2}\vec{H}$ suffices as our choice of timelike unit normal.

Since $\langle D_V \vec{H}, P\rangle = V\langle \vec{H}, P\rangle = 0$ and $\langle D_V \vec{H}, \vec{H}\rangle = \frac{1}{2}V\langle \vec{H}, \vec{H}\rangle = 0$, we conclude that $D_V^\perp \vec{H} = 0$. From the Codazzi equation ([21],pg.115)

$$(D_V^\perp \,\mathrm{II})(W,U) = (D_W^\perp \,\mathrm{II})(V,U)$$

the fact that $D^\perp \vec{H} = 0$ implies that $(\nabla_V \hat{a})(W,U) = (\nabla_W \hat{a})(V,U)$ for ∇ the induced connection on Σ. As a result, taking a trace over V, U and using the fact that contraction commutes with covariant differentiation, we have

$$(\nabla \cdot \hat{a})(W) = \mathrm{tr}_\Sigma(\nabla_W \hat{a}) = W\,\mathrm{tr}_\Sigma\,\hat{a} = 0.$$

It is a well known consequence of the Uniformization Theorem that the divergence operator on a spacelike 2-sphere is injective when restricted to symmetric tracefree 2-tensors (see for example, [25]), so we conclude that $\hat{a} = 0$ which implies $\hat{\mathrm{II}} = 0$. From this it follows that $\langle D_V \vec{H}, W\rangle = -\langle \vec{H}, \mathrm{II}(V,W)\rangle = -\frac{1}{2}\langle \vec{H}, \vec{H}\rangle\langle V, W\rangle = -2\langle V, W\rangle$ and therefore $D_V \vec{H} = -2V$. Finally, this gives us that

$$D_V N = D_V P + \frac{1}{2} D_V \vec{H} = V - V = 0,$$

as desired. □

From Lemma 1.1.1, the one-to-one correspondence between boosts $\phi_{\vec{v}}$ and round foliations of Ω is therefore evident via (1.3).

1.3 Formulations of the Penrose conjecture

An immediate difficulty we face when we delve into the general theory is to understand the nature and behavior of energy and momentum, and, by extension, a meaningful measurement of mass. Fortunately, a means of measuring total energy and momentum is afforded by certain spacetimes.

For matter that is isolated and locally concentrated within a spacetime $(\mathcal{M}, g)$, the 'farther away' one is from the matter content the more the geometry settles towards 'flatness'. With regards to the metric g, this translates into certain decay conditions on the associated curvature tensors, and such spacetimes are called *asymptotically flat.*

1.3.1 The spacelike setting

A hypersurface $\mathcal{N}^3 \subset \mathcal{M}$ with induced Riemannian metric $\bar{g}$ is said to be *asymptotically Euclidean* if $\bar{g}$ decays sufficiently fast to the flat metric δ on $\mathbb{R}^3$ and, up to a compact region $\mathcal{C}$ (surrounding the matter content), the sub-structure of $\mathcal{N}-\mathcal{C}$ is also in agreement with (possibly more than one copy of) $\mathbb{R}^3$. Specifically, $\mathcal{N}-\mathcal{C} = \bigcup_{i=1}^{k} \mathcal{N}_i$ where each *end* $\mathcal{N}_i$ is diffeomorphic to $\mathbb{R} - B^3$, for B^3 the unit ball. Moreover, in the chart $\{x_i\}$ for $\mathbb{R}^3$ (with radius $r^2 := \sum_{i=1}^{3} x_i^2$) we have that $\bar{g}_{ij} = \delta_{ij} + \epsilon_{ij}$ and $r|\epsilon_{ij}| + r^2(|\partial\epsilon_{ij}| + |K_{ij}|) + r^3(|\partial\partial\epsilon_{ij}| + |\partial K_{ij}|) \leqslant C$, where K is the second fundamental form of $\mathcal{N}$. The pair $\mathcal{N} \subset \mathcal{M}$ therefore become asymptotically characteristic of a Euclidean slice $\mathbb{E} \subset \mathbb{R}_1^4$. Within these isolated systems, similarly to a particle in $\mathbb{R}_1^4$, we observe an abstract ADM energy-momentum $(E, \vec{P})$ ([2]) that is given (assuming summation over repeated indices) by

$$E = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} (\partial_j \bar{g}_{ij} - \partial_i \bar{g}_{jj}) \vec{n}_i d\sigma \tag{1.5}$$

and

$$P^i = \lim_{r\to\infty} \frac{1}{8\pi} \int_{S_r} (K^i_j - \delta^i_j K_{kk}) \vec{n}_j d\sigma \tag{1.6}$$

where S_r represents a coordinate sphere of radius r with unit normal $\vec{n}$. It can be shown that E and P^i do not depend on our choice of chart $\{x_i\}$ ([3]), and two distinct asymptotically Euclidean slicings of $\mathcal{M}$ differ by a Minkowskian boost (1.2) in their respective measurements of $(E, \vec{P})$ (see [24]).

We are now in a position to state a beautiful result first proved by Schoen and Yau using minimal surface techniques ([26],[27]) and then by Witten using spinors

([32]).

Theorem 1.3.1 (Positive Mass Theorem)**.** *Let $(\mathcal{N}, \bar{g}) \hookrightarrow (\mathcal{M}, g)$ be an asymptotically Euclidean hypersurface, where the metric g satisfies the Dominant Energy Condition. Then for each end $\mathcal{N}_i$ we have that*

$$E^2 \geqslant |\vec{P}|^2.$$

Moreover, if $E = 0$ for some i, then $(\mathcal{N}, \bar{g}) = (\mathbb{R}^3, \delta)$.

It follows therefore that the mass of the spacetime satisfies

$$M = \sqrt{E^2 - |\vec{P}|^2} \geqslant 0$$

as expected. The Penrose conjecture further predicts a stricter lower bound for the total mass M whenever spacetimes contain any black hole horizons, namely, the inequality (1.1).

A natural special case to consider is when the slice $(\mathcal{N}, \bar{g})$ represents a rest-frame $P^i = 0$, $E = M$. We see from (1.6) that this holds in the event that $\mathcal{N}$ is *totally geodesic*, i.e. $K = 0$. Since the tensor K is given by a normal variation of $\bar{g}$ off of $\mathcal{N}$ (by first variation of area) we would obtain this condition from a time-symmetric slicing of the spacetime $\mathcal{M}$. From the Gauss equation, it then follows that the Dominant Energy Condition is equivalent to the statement that $\mathcal{N}$ have non-negative scalar curvature $R_{\bar{g}} \geqslant 0$. It is also known in this setting that the black hole horizon is represented by an 'outermost' minimal surface Σ_0 possibly with multiple components; each component a sphere and no two spheres intersecting (see [20, 15]). From this the Penrose conjecture completely reduces to a statement relating the total mass of the time-symmetric three-dimensional Riemannian manifold $\mathcal{N}$ and the area of $\Sigma_0 \hookrightarrow \mathcal{N}$, called the *Riemannian Penrose Inequality* (or RPI). A fundamental breakthrough came at the very end of the twentieth century with a complete proof of the RPI. Whenever Σ_0 is a single connected component Huisken and Ilmanen proved the RPI in 1997 ([15]) which was subsequently generalized to multiple components by Bray in 1999 ([6]) using a completely new approach.

Theorem 1.3.2. *(Bray) Suppose $(\mathcal{N}, \bar{g})$ is complete, has non-negative scalar curvature and contains an outermost minimal surface $\Sigma_0 = \bigcup_{i=1}^n S_i$. Then*

$$M \geqslant \sqrt{\frac{\sum_i |S_i|}{16\pi}}$$

with equality only in the case that $(\mathcal{N}, \bar{g})$ is isometric to a time-symmetric slice of the Schwarzschild spacetime (see Section 1.4.1 below) of mass M.

An interesting energy functional for any closed spacelike surface Σ introduced by Hawking ([12]) is defined by

$$E_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\Big(1 - \frac{1}{16\pi}\int_\Sigma \langle \vec{H}, \vec{H} \rangle dA\Big)$$

for $\vec{H}$ the mean curvature of Σ. This *Hawking Energy* was the definitive tool used by Huisken and Ilmanen to prove the Riemannian Penrose Inequality. An observation due to Geroch ([10]) shows that under inverse mean curvature flow, E_H is non-decreasing within slices of non-negative scalar curvature. It also follows for coordinate spheres S_r of an asymptotically Euclidean slice that $\lim\limits_{r\to\infty} E_H(S_r) = E$. So if the slice $\mathcal{N}$ is time-symmetric, not only do we conclude that $\lim\limits_{r\to\infty} E_H(S_r) = M$ but on a minimal surface Σ_0 (where $\vec{H} = 0$) it also follows that $E_H(\Sigma_0) = \sqrt{\frac{|\Sigma_0|}{16\pi}}$. From Geroch's observations, Jang and Wald ([16]) observed a potential application towards the Riemannian Penrose Inequality given the existence of a smooth flow from the vicinity of a horizon to round spheres at infinity. It was soon realized however that this flow would not remain smooth in general, which is where Huisken and Ilmanen's work was needed to close the argument. They were able to show the existence of a weakly defined inverse mean curvature flow to overcome possible singularities and found that the leaves Σ_s approached round spheres asymptotically as needed. Unfortunately, the monotonicity of the flow is dependent on the topology of the initial surface, requiring that the flow begin on a connected component of $\Sigma_0 = \bigcup_i S_i$. As a result, they were able to show

$$\max_i \sqrt{\frac{|S_i|}{16\pi}} = E_H(0) \leqslant \lim_{s\to\infty} E_H(s) = M.$$

Rather than a flow of surfaces within the fixed geometry of the slice, Bray's insight was instead to construct a flow of metrics that vary the geometry of the slice via a one-parameter family of conformal factors. Bray was able to construct this flow so that the area of the outermost minimal surface was unchanging, the scalar curvature remained non-negative (enforcing $M(t) \geqslant 0$ by the Positive Mass Theorem) and the ADM mass was non-increasing. Moreover, the flow approaches spherical symmetry, so combined with positive mass this ensures the limit is a time-symmetric slice of the Schwarzschild spacetime. Since (1.1) achieves equality for these slices in Schwarzschild we conclude that:

$$\sqrt{\frac{|\Sigma_0|}{16\pi}} = \lim_{t\to\infty} \sqrt{\frac{|\Sigma_t|}{16\pi}} = \lim_{t\to\infty} M(t) \leqslant M.$$

Where Huisken and Ilmanen's proof is strongly dependent on the dimension of the slice, Bray's argument was shown by Bray and Lee ([8]) to generalize to dimensions less than eight. In either instance the case of equality for (1.1) enforces $\mathcal{N}$ to be the time-symmetric slice of Schwarzschild.

1.3.2 The null setting

In Minkowski spacetime, upon changing our coordinates to 'ingoing null coordinates' $(t,r) \to (v,r)$ where $v = t + r$, the associated quadratic form is given by

$$ds^2 = -dv^2 + 2dvdr + r^2(d\vartheta^2 + (\sin\vartheta)^2 d\varphi^2),$$

and we recognize the past null cones along the time-axis as the slices $\Omega = \{v = v_0\}$. We find that $\partial_v = \partial_t$ and the gradient of v satisfies $Dv = \partial_r \in \Gamma(T\Omega) \cap \Gamma(T^\perp\Omega)$, which is a past-pointing null vector. From the identity $D_{Df}Df = \frac{1}{2}D|Df|^2$, we conclude that integral curves of ∂_r are geodesic, so ∂_r generates the light rays that rule Ω. In fact, any spherical cross-section $\Sigma \hookrightarrow \Omega$ is uniquely identified as a graph over $\mathbb{S}^2$ by specifying $r|_\Sigma$. For a cross-section $\Sigma := \{r = \omega(\vartheta,\varphi)\}$, we may then assign a normal null basis $\{\underline{L}, L\} \in T^\perp\Sigma$ such that $\underline{L} = \partial_r$ and $\langle\partial_r, L\rangle = 2$, from which we decompose the second fundamental form II of Σ accordingly to $\underline{\chi} := -\langle \mathrm{II}, \partial_r\rangle$ and $\chi := -\langle \mathrm{II}, L\rangle$. For Σ, as shown in Section 6.1 (with $\beta = M = 0$), denoting by $\nabla\!\!\!/$ the induced connection, $\zeta(V) := \frac{1}{2}\langle D_V\partial_r, L\rangle$ the *connection 1-form* for the normal bundle, and $\mathring{\gamma}$ the standard round metric on $\mathbb{S}^2$, one has

$$\begin{aligned} \gamma &= \omega^2\mathring{\gamma}, & \underline{\chi} &= \frac{1}{\omega}\gamma, & \chi &= \frac{1}{\omega}(1 + |\nabla\!\!\!/\omega|^2)\gamma - 2H^\omega, \\ \zeta &= -d\!\!\!/\log\omega, & \operatorname{tr}\underline{\chi} &= \frac{2}{\omega}, & \operatorname{tr}\chi &= \frac{2}{\omega}(1 - \mathring{\Delta}\log\omega). \end{aligned}$$

Therefore, for any geodesically induced foliation of Ω along $\partial_s = \psi\partial_r$ (for some $\psi \in \mathcal{F}(\mathbb{S}^2)$), dependence of the data on the affine parameter s is given via $\omega(s,\vartheta,\varphi) = s\psi(\vartheta,\varphi)$. According to the null basis $\{\psi\partial_r, \frac{1}{\psi}L\}$ we therefore get

$$\begin{aligned} \gamma_s &= s^2\psi^2\mathring{\gamma}, & \underline{\chi}_s &= \psi\underline{\chi} = \frac{1}{s}\gamma, & & \\ \zeta_s &= \zeta + d\!\!\!/\log\psi = 0, & \operatorname{tr}\underline{\chi}_s &= \psi\operatorname{tr}\underline{\chi} = \frac{2}{s}, & \operatorname{tr}\chi_s &= \frac{\operatorname{tr}\chi}{\psi} = \frac{2}{s}\mathcal{K}_{\psi^2\mathring{\gamma}}. \end{aligned}$$

For asymptotically flat spactimes, one obtains the notion of an *asymptotically flat nullcone* Ω by specifying decay along a geodesic foliation that mirrors the above dependence on the affine parameter s to leading order (see Section 5.3 for precise definitions). The work of Mars and Soria ([18]) then shows that $\lim_{s_\star \to \infty} E_H(\Sigma_{s_\star}) < \infty$ for any asymptotically geodesic foliation $\{\Sigma_{s_\star}\}$ satisfying $s = \psi s_\star + \xi$, where ξ decays 'sufficiently fast'. As was the case for asymptotically Euclidean slices, we find that $E_H(\Sigma_{s_\star})$ approaches a measure of energy called the Bondi-energy E_B if the leaves of our foliation $\{\Sigma_{s_\star}\}$ approach asymptotically round spheres. This coupling between energy measurements and asymptotically round foliations of Ω should be familiar given our analysis of the Minkowski spacetime, where we showed a one-to-one correspondence between the boosts (1.2) and round foliations of the past null cone of a point. Heuristically, we imagine an asymptotically round foliation of Ω induced by intersection with an asymptotically Euclidean slicing of $\mathcal{M}$ (see Figure 1.2 below). As these intersections asymptotically coincide with coordinate spheres in the slicing (as is the case in Minkowski spacetime) the Hawking energy becomes comparable to the energy associated with the slicing. Minimizing over all possible energies, we therefore obtain the Bondi-mass m_B. By taking a Riemannian hypersurface asymptotic to Ω, Schoen and Yau (see [28]) were able to construct an asymptotically Euclidean manifold (not necessarily embeddable in $\mathcal{M}$) whose ADM mass is no larger than m_B. From this the positivity of the Bondi-mass follows from the Positive Mass Theorem.

In order to formulate the Penrose conjecture in this setting, we need to first motivate a definition of a horizon Σ_0. For any cross-section $\Sigma \hookrightarrow \Omega$, it follows that we recover Ω 'outside Σ' by emitting null geodesics along our past-pointing null geodesic generator $\underline{L} \in \Gamma(T\Omega) \cap \Gamma(T^\perp\Omega)$. If Ω is asymptotically flat, we also conclude that any foliation along $\underline{L}$ has expanding area (see Lemma 5.3.1), so $\underline{L}$ points 'outwards and to the past' of Σ. It follows, by rescaling $\underline{L}$, that Σ admits a null basis $\{L^- = \frac{\underline{L}}{\operatorname{tr}\underline{\chi}}, L^+ = \operatorname{tr}\underline{\chi} L\}$ such that $\langle L^-, L^+\rangle = 2$ (i.e. L^+ is 'outward and to the future' of Σ) and therefore $-2\vec{H} = L^+ - \langle \vec{H}, L^+\rangle L^-$. For an infinitesimal flow

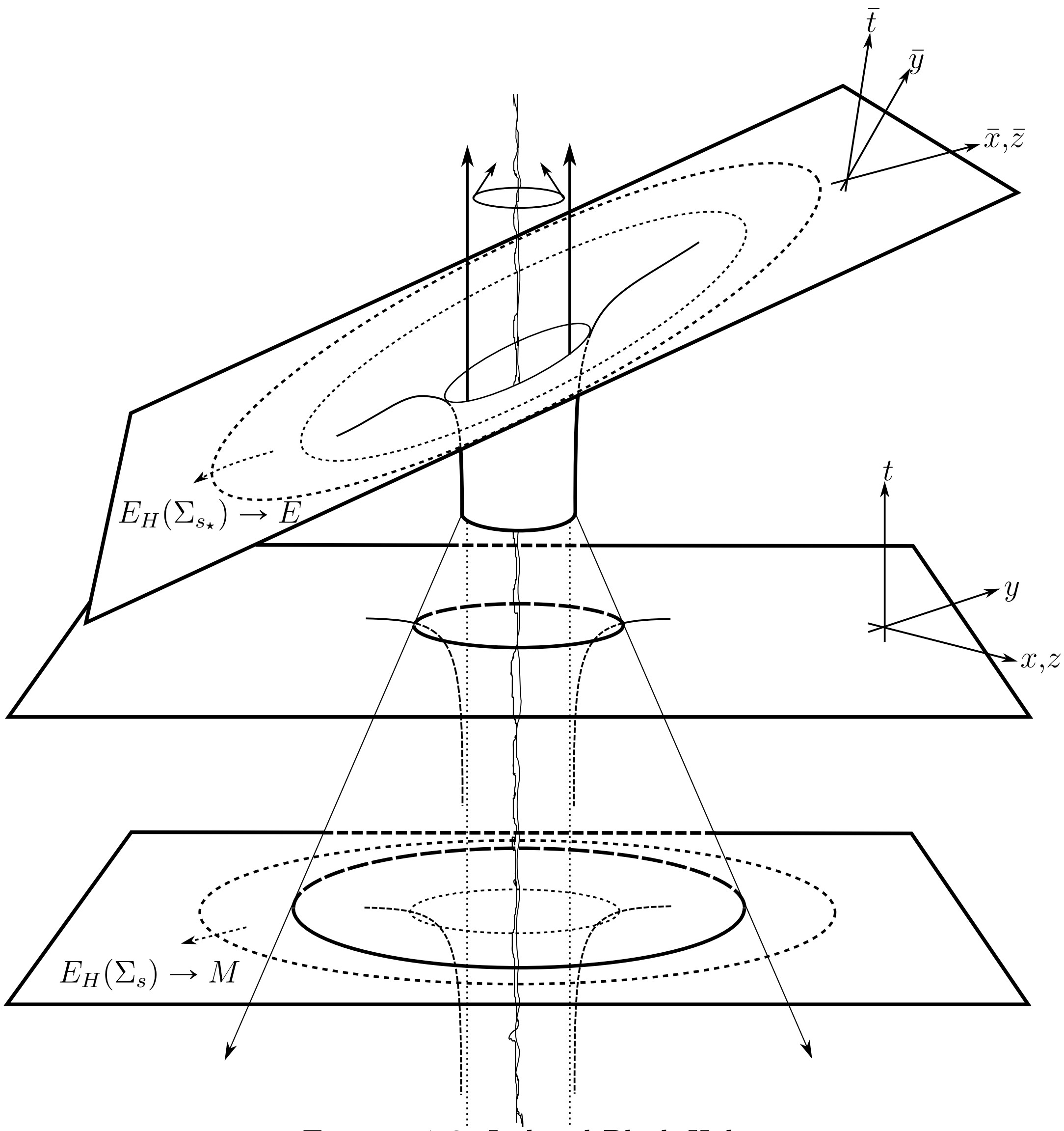

FIGURE 1.2: Isolated Black Hole

along L^+ off of our initial surface Σ, we have from a first variation of the area form that $\dot{dA} = -\langle \vec{H}, L^+ \rangle dA = \langle \vec{H}, \vec{H} \rangle dA$. Thus, if we're able to locate a $\Sigma \hookrightarrow \Omega$ for which $\vec{H}$ is a timelike vector, then Σ is 'trapped' in the sense that light rays emitted off of the surface of Σ yield a collapsing sphere. A horizon is therefore identified by a *marginally outer trapped surface* Σ_0 where the mean curvature $\vec{H}$ is null, i.e. $\langle \vec{H}, \vec{H} \rangle = 0$.

We notice immediately for a horizon $\Sigma_0 \hookrightarrow \Omega$ that $E_H(\Sigma_0) = \sqrt{\frac{|\Sigma_0|}{16\pi}}$. As was the case for Huisken and Ilmanen's proof of the RPI, we ask whether we're able to interpolate the Hawking energy along Ω from the horizon to *null infinity*. In his PhD thesis, Johannes Sauter ([25]) showed that, for a shear free nullcone Ω (i.e. $\hat{\underline{\chi}} = 0$) and $\mathcal{M}$ a vacuum spacetime, one is able to solve a system of ODEs to yield explicitly the data on Ω. This then enables a direct analysis of E_H at null infinity that allowed Sauter to prove the Penrose conjecture in this setting. From an observation attributed to Christodoulou (see [25]), monotonicity of E_H follows for foliations $\{\Sigma_{s_\star}\}$ if either the *mass aspect function* $\mu := \mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H}\rangle - \nabla\!\!\!/ \cdot \zeta$ or $\operatorname{tr}\underline{\chi}$ remain constant on each $\Sigma_{s_\star}$. Interestingly, the latter flow is in fact a null 'inverse mean curvature' flow since the flow vector $\underline{L}$ satisfies $-\langle \underline{L}, \vec{H}\rangle = \operatorname{tr}\underline{\chi} = \mathit{const.}$ on each $\Sigma_{s_\star}$. Sauter was able to show that, for small pertubations of Ω off of the shear free condition, one obtains global existence of either flow and that E_H converges. Unfortunately though, unlike the Huisken and Ilmanen case, one is unable to conclude that the foliating 2-spheres become round asymptotically. In fact, Bergqvist ([4]) noticed this exact difficulty had been overlooked in an earlier work of Ludvigsen and Vickers ([17]) towards proving the *weak null Penrose conjecture*, namely $\sqrt{\frac{|\Sigma_0|}{16\pi}} \leqslant E_B$.

Alexakis ([1]) was able to prove the null Penrose conjecture for vacuum perturbations of the black hole exterior in Schwarzschild spacetime by successfully using the monotonicity of E_H along the null inverse mean curvature flow. Here the author was once again aided by an explicit analysis of E_H at null infinity. Work by Mars and Soria ([18]) followed soon after that identified the necessary conditions on Ω (inside general ambient spacetimes) needed to free up an explicit analysis of $\lim\limits_{s\to\infty} E_H(\Sigma_s)$ along geodesic foliations. With their notion of an asymptotically flat null hypersurface Ω, the authors were able to show an explicit limit of E_H at null infinity along various asymptotically geodesic foliations. In a later work ([19]), Mars and Soria constructed a new functional on 2-spheres and showed for a special foliation $\{\Sigma_\lambda\}$ off of the horizon Σ_0 called *geodesic asymptotically Bondi* (or GAB) that, $\sqrt{\frac{|\Sigma_0|}{16\pi}} \leqslant \lim_{\lambda\to\infty} E_H(\Sigma_\lambda) < \infty$. Thus, for GAB foliations that approach round spheres, the authors reproduce the weak null Penrose conjecture of Bergqvist ([4]), Ludvigsen and Vickers ([17]). Unfortunately, as in the aforementioned work of Sauter, Bergqvist, Ludvigsen and Vickers there is no guarantee of asymptotic roundness.

1.4 Schwarzschild Spacetime

Relaxing our restriction from flat to spherically symmetric solutions in vacuum ($G =\ 0$ or, equivalently $Ric = 0$), we extend beyond Minkowski to the 1-parameter family of *Schwarzschild solutions*. Modeling a static black hole of mass M, the Schwarzschild spacetime is characterized in spherical polar coordinates by the quadratic form

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})} + r^2(d\vartheta^2 + (\sin\vartheta)^2 d\varphi^2)$$

valid for $2M > r > 0$, $r > 2M$. From this we confirm that $M = 0$ reproduces the quadratic form of Minkowski spacetime. The maximal extension of this geometry is called the Kruskal spacetime $(\mathbb{P} \times_r \mathbb{S}^2, g_K)$ which is given by the warped product of the Kruskal Plane $\mathbb{P} := \{UV > -2Me^{-1}\}$ and the standard round $\mathbb{S}^2$ with warping function $r = g^{-1}(UV)$ for $g(r) = (r - 2M)e^{\frac{r}{2M}-1}$, $r > 0$. Therefore, the associated quadratic form that extends the Schwarzschild metric is given by:

$$ds^2 = 2F(r)dVdU + r^2(d\vartheta^2 + (\sin\vartheta)^2 d\varphi^2)$$

where $F(r) = \frac{8M^2}{r}e^{1-\frac{r}{2M}}$. We recover the Schwarzschild spacetime on $V > 0$, $U \neq 0$ with the coordinate change $t = 2M \log|\frac{V}{U}|$ ([21]). Each round $\mathbb{S}^2$ has area $4\pi r^2$ so we interpret r as a 'radius function', the curvature singularity at $r = 0$ we trace back to the function $F(r)$, known as the 'black hole' singularity.

1.4.1 *Time-symmetric slices of Schwarzschild*

The time-symmetric slices of the Riemannian Penrose inequality are given by $\mathcal{N} := \{V = e^{\frac{t_0}{2M}} U\}$ or $t = t_0$ in the Schwarzschild region. The fact that $\mathcal{N}$ is asymptotically Euclidean is evident under a change to *isotropic coordinates* represented by $(t, r) \to (t, R)$ for $r = R(1 + \frac{M}{2R})^2$. The metric $\bar{g}$ is represented by:

$$\frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\vartheta^2 + (\sin\vartheta)^2 d\varphi^2) = \Big(1 + \frac{M}{2R}\Big)^4 (dx^2 + dy^2 + dz^2)$$

where $R^2 = x^2 + y^2 + z^2$. We see that the coordinate singularity at $r = 2M$ has been removed in the change to isotropic coordinates. Moreover, for the diffeomorphism $\phi : (0, \infty) \to (0, \infty)$ given by $\phi(R) = (\frac{M}{2})^2 \frac{1}{R}$ inspection of isotropic coordinates in spherical polar form identifies an isometry of $\mathcal{N}$. One can also show that ϕ is the restriction to $\mathcal{N}$ of the isometry given by $U \to -U, V \to -V$. From this we conclude that $\mathcal{N}$ has two asymptotically Euclidean ends joined at the minimal sphere $R = \frac{M}{2}$.

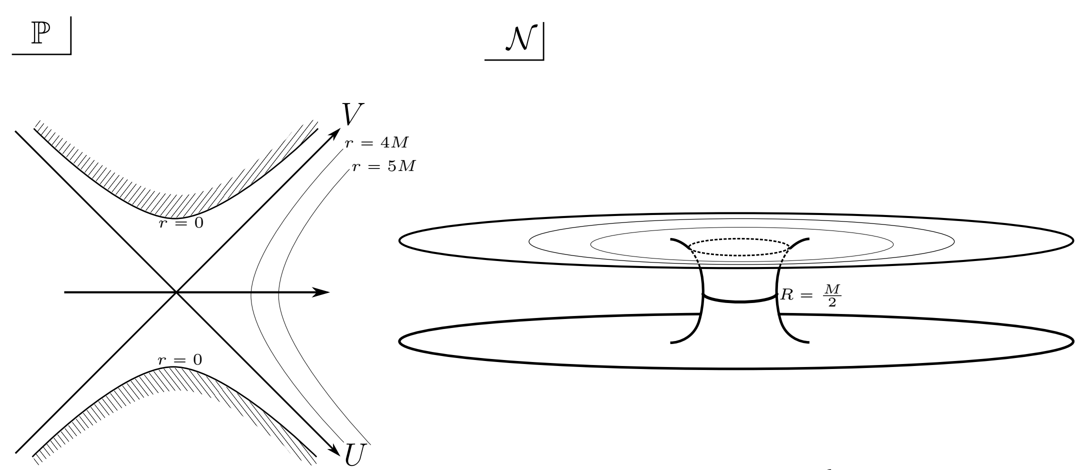

FIGURE 1.3: The time-symmetric slice $\mathcal{N}$

This minimal sphere corresponds to the black hole horizon at $r = 2M$. We also show that the second fundamental form K vanishes. Since the components of the Schwarzschild metric are independent of the coordinate t, we know ∂_t is a Killing vector i.e. $\langle D_X\partial_t, Y\rangle + \langle D_Y\partial_t, X\rangle = 0$ for all vector fields X, Y. So restricting to $X, Y \in \Gamma(T\mathcal{N})$ we conclude that

$$K(X,Y) \propto -\langle\partial_t, D_XY\rangle = -\frac{1}{2}\Big(\langle\partial_t, D_XY\rangle + \langle\partial_t, D_YX\rangle + \langle\partial_t, [X,Y]\rangle\Big)$$
$$= \frac{1}{2}\Big(\langle D_X\partial_t, Y\rangle + \langle D_Y\partial_t, X\rangle\Big) = 0$$

as expected.

In the Appendix, we further explicitly analyze the effect on Bartnik data for coordinate spheres of a time-symmetric slice that undergoes a boost of its asymptotic frame of reference (recall Figure 1.2). As a consequence, we expand calculations of Wang and Yau ([31]) in obtaining the energy-momentum (according to E_H and (1.6)). Specifically, we show, under a boost of isotropic coordinates

$$\begin{pmatrix} t \\ z \end{pmatrix} \to \begin{pmatrix} \cosh\psi & -\sinh\psi \\ -\sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} t \\ z \end{pmatrix}$$

(here ψ represents a constant 'rapidity'), that

$$(E, \vec{P}) : (M, 0, 0, 0) \to (M\cosh\psi, 0, 0, M\sinh\psi).$$

1.4.2 Standard Nullcones of Schwarzschild

The past nullcone of a point in Minkowski spacetime also has a counterpart in Schwarzschild spacetime. In order to show this, we make yet another change of coordinates to the so called *ingoing Eddington-Finkelstein* coordinates, $(t, r) \to (v, r)$ whereby $dv = dt + \frac{dr}{1-\frac{2M}{r}}$:

$$ds^2 = -(1 - \frac{2M}{r})dv^2 + 2dvdr + r^2(d\vartheta^2 + (\sin\vartheta)^2 d\varphi^2).$$

Once again, setting $M = 0$ the resemblance with Minkowski is apparent. The *standard nullcone of Schwarzschild* is therefore given by slices of the form $\Omega := \{v = v_0\}$.

Up to a positive constant multiple we find that $V = e^{\frac{v}{4M}}$, so that the coordinates $(v, r, \vartheta, \varphi)$ cover the whole Schwarzschild region $V > 0$, including the horizon at $U = 0$. Arguing identically as in the case of Minkowski, we find our null geodesic generator to be $\partial_r = \frac{\partial_U}{F(r)}$, and any cross section $\Sigma \hookrightarrow \Omega$ is uniquely identified by specifying $r|_\Sigma = \omega(\vartheta, \varphi)$.

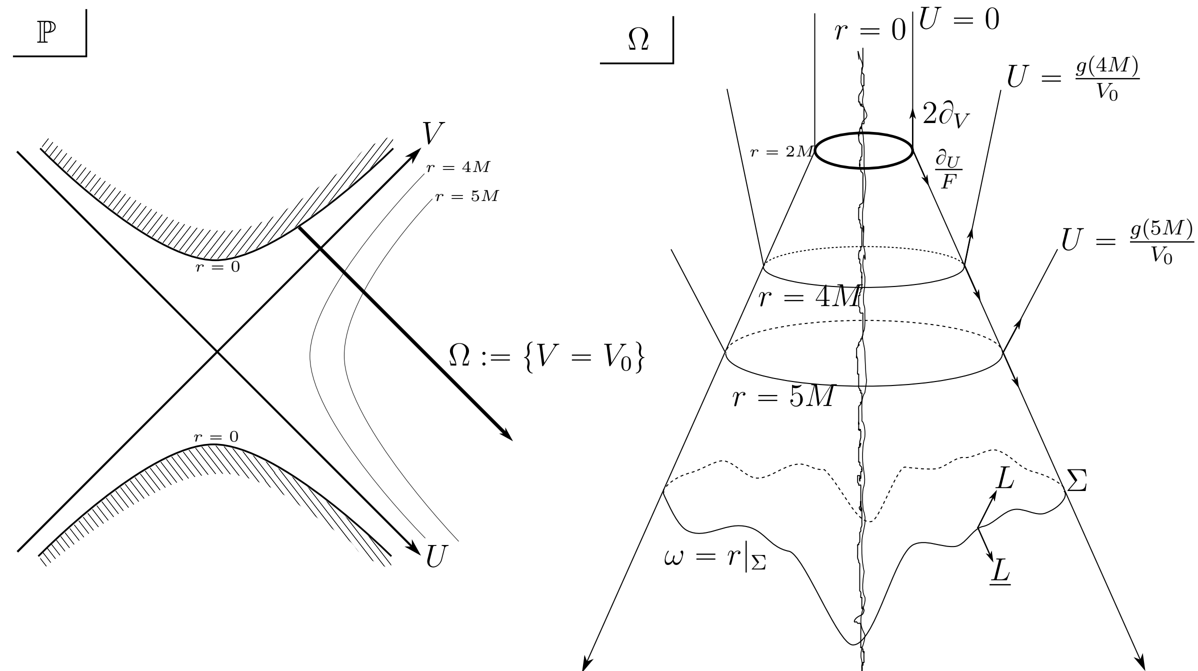

FIGURE 1.4: Standard Nullcone of Schwarzschild Ω

Endowed with a null normal basis $\{\underline{L} = \partial_r, L\} \subset T^\perp\Sigma$ such that $\langle \partial_r, L\rangle = 2$, the data on Σ is found in Section 3.2 (or Section 6.1 with $\beta = 0$)

$$\begin{aligned} &\gamma = \omega^2 \mathring{\gamma}, && \underline{\chi} = \frac{1}{\omega}\gamma, && \chi = \frac{1}{\omega}\Big(1 - \frac{2M}{\omega} + |\nabla\!\!\!/\,\omega|^2\Big)\gamma - 2H^\omega, \\ &\zeta = -d\!\!\!/ \log\omega, && \operatorname{tr}\underline{\chi} = \frac{2}{\omega}, && \operatorname{tr}\chi = \frac{2}{\omega}\Big(1 - \frac{2M}{\omega} - \mathring{\Delta}\log\omega\Big). \end{aligned}$$

It follows that $\operatorname{tr}\underline{\chi}\operatorname{tr}\chi = \langle \vec{H}, \vec{H}\rangle = \frac{4}{\omega^2}(1 - \frac{2M}{\omega} - \mathring{\Delta}\log\omega)$ and, by the Maximum Principle, the cross-section of Ω given by $\Sigma_0 := \{r = 2M\}$ uniquely satisfies $\langle \vec{H}, \vec{H}\rangle = 0$. So Σ_0 is a horizon, in fact, from either the ingoing Eddington-Finkelstein or

Kruskal metric, we see that $U = 0$ if and only if $r = 2M$ is a null hypersurface with spherical cross-sections, all of area $16\pi M^2$. This hypersurface represents the outermost boundary of the black hole singularity called the *event horizon*. Regarding asymptotic flatness, from a geodesic foliation $\Sigma_s := \{r = s\psi(\vartheta, \varphi)\}$ of Ω we see that the only distinction from the case in Minkowski spacetime is higher order decay in the datum $\operatorname{tr}\chi_s = \frac{2}{s}\mathcal{K}_{\psi^2\mathring{\gamma}} - \frac{1}{s^2}\frac{2M}{\psi^2}$. In fact, for any cross-section Σ we observe that $\mathcal{K} - \frac{1}{4}\langle\vec{H}, \vec{H}\rangle = \frac{2M}{\omega^3}$. Thus, from the Gauss-Bonnet Theorem followed by Jensen's inequality, we conclude, as did Sauter (see [25]), that,

$$E_H(\Sigma) = \sqrt{\frac{\int \omega^2 d\sigma}{16\pi}} \int \frac{2M}{\omega} d\sigma \geqslant M,$$

where integration is over the standard round $\mathbb{S}^2$. We have equality only if $\omega(\vartheta, \varphi) = \omega_0$, which is given by intersections of the time-symmetric slices $\mathcal{N}$ with Ω (or a rest frame measurement with respect to our black hole). Moreover, for round spheres given by (1.3), we see $E_H(\Sigma_{s\psi}) = \frac{M}{\sqrt{1-|\vec{v}|^2}}$, in exact agreement with the boosted energies of a particle (here the black hole) in special relativity.

1.5 Mass rather than Energy

Although the Hawking Energy enjoys monotonicity and convergence along certain flows, difficulty remains in assigning physical significance to the convergence of E_H due to the lack of control on the asymptotics of such flows. We expect these difficulties may very well be symptomatic of the fact that an energy functional is particularly susceptible to the plethora of ways boosts can develop along any given flow.

Analogous to the addition of 4-velocities in special relativity, $P_1 + P_2 = P_3$ which gives $E_3 = E_1 + E_2$ (see Figure 1.5), we expect an infinitesimal null flow of Σ within a fixed reference frame to raise energy due to an influx of matter. However, with no a priori knowledge of the flow, we have no way to fix or even identify a reference frame.

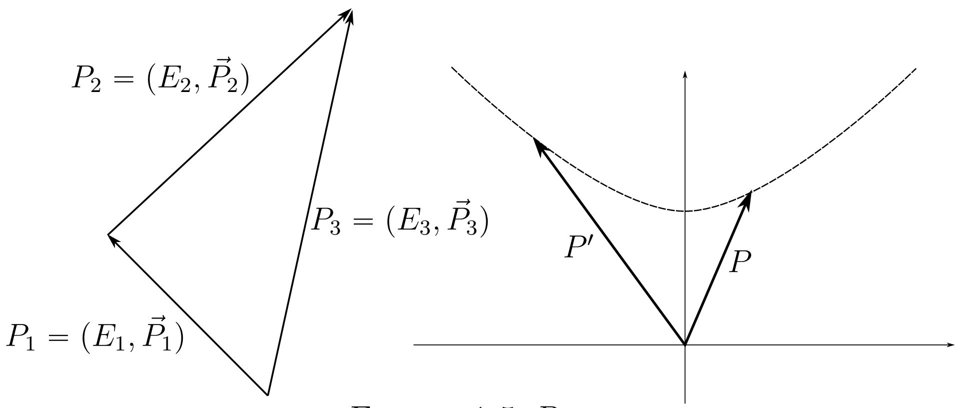

FIGURE 1.5: Boosts

So it is likely that 'phantom energy' will accumulate from infinitesimal boosts along the flow, in analogy with special relativistic boosts, $P \to P'$ (i.e energy increases), or $P' \to P$ (i.e. energy decreases) as shown in Figure 1.5. Geometrically, we expect this to manifest along the flow in a (local) 'tilting' of Σ. One may even expect a net decrease in energy. For the standard nullcone Ω of Schwarzschild spacetime, if we insist upon the use of E_H, we observe as in the previous section that our only choice of foliation increasing to the mass M is to foliate Ω with time-symmetric slice intersections (i.e. $\omega = const.$). Not only is this flow highly specialized, it dictates strong restrictions on our initial choice of Σ from which to begin the flow. This is to be expected of a quasi-local energy due to its inherent sensitivity to boosts of our abstract reference frame along the flow.

This is not a problem, however, if we appeal instead to mass rather than energy since boosts leave mass invariant, $M^2 = E^2 - |\vec{P}|^2 = (E')^2 - |\vec{P}'|^2 = (M')^2$. Moreover, by virtue of the Lorentzian triangle inequality (provided all vectors are timelike and either all future or all past-pointing), along any given flow the mass should always increase:

$$M_3 = |(E_1 + E_2, \vec{P}_1 + \vec{P}_2)| \geqslant |(E_1, \vec{P}_1)| + |(E_2, \vec{P}_2)| = M_1 + M_2.$$

We hope therefore that, by appealing instead to a quasi local mass functional, a larger class of valid flows and more generic monotonicity should arise. In this thesis, we construct such a mass functional by first finding an optimal choice of flux function for E_H.

1.5.1 *Overview of Thesis*

In Chapter 2, we develop the necessary technical preliminaries in order to provide a canonical choice of null frame $\{L^-, L^+\} \in T^\perp\Sigma$ for 2-spheres Σ in spacetime. From this, we introduce a new flux function ρ and use it to construct a new quasi-local mass $m(\Sigma)$. We also state and provide short discussions of our two main results, Theorem 2.1.1 and Theorem 2.1.2, motivating the notion of a (P) (or (SP))-foliation of a nullcone.

In Chapter 3 we motivate our construction of ρ and $m(\Sigma)$ from some of our simplest example spacetimes. Specifically, in Section 3.1, we identify an interesting relationship between the canonically defined connection 1-form τ of $T^\perp\Sigma$, the shear tensor $\hat{\chi}^-$, and the underlying null geometry of Σ in a space form $\mathcal{M}$. By way of the Gauss equation, this yields a simple expression for our flux function ρ and mass $m(\Sigma)$. Moreover, revisiting the standard nullcone of Schwarzschild, we show that $m(\Sigma)$ is chosen precisely to yield the mass M of the black hole irrespective of the choice of cross-section Σ.

In Chapter 4 we expand to general spacetimes and calculate the propagation of ρ along an arbitrary foliation of a nullcone Ω. From this, we prove Theorem 2.1.1, which indicates fairly generic monotonicity of our mass functional $m(\Sigma)$. Specifically, we have monotonicity along any (P) (or (SP))-foliation.

In Chapter 5, we decompose the flux ρ of any cross-section $\Sigma \hookrightarrow \Omega$ in terms of the data for some background foliation of our nullcone Ω. We then introduce the necessary asymptotics to explicitly formulate the limit of our mass functional $m(\Sigma)$. We find a new notion of mass for Ω, namely $M := \lim_{s\to\infty} m(\Sigma_s)$, which is completely independent of our choice of asymptotically geodesic foliation. This allows us to prove Theorem 2.1.2, yielding the Null Penrose Inequality under fairly generic conditions.

In Chapter 6, we investigate spherically symmetric spacetimes and identify a class of perturbations of the black hole exterior admitting asymptotically flat nullcones of strong flux decay with an (SP)-foliation. As a result, the existence of such perturbations satisfying the dominant energy condition gives the Null Penrose Inequality (1.1).

2

Technical Background

2.1 Preliminaries and Main Results

A spacetime $(\mathcal{M}, g)$ is defined to be a four dimensional smooth manifold $\mathcal{M}$ equipped with a metric $g(\cdot,\cdot)$ (or $\langle\cdot,\cdot\rangle$) of Lorentzian signature $(-,+,+,+)$. We assume that the spacetime is both orientable and time orientable, i.e. admits a nowhere vanishing timelike vector field, defined to be future-pointing.

Throughout this paper, we will denote by Σ a spacelike embedding of a sphere in $\mathcal{M}$ with induced metric γ. It is well known that Σ has trivial normal bundle $T^{\perp}\Sigma$ with induced metric of signature $(-,+)$. From any choice of null section $\underline{L} \in \Gamma(T^{\perp}\Sigma)$, we have a unique null partner section $L \in \Gamma(T^{\perp}\Sigma)$ satisfying $\langle \underline{L}, L\rangle = 2$, providing $T^{\perp}\Sigma$ with a null basis $\{L, \underline{L}\}$. We also notice that any 'boost' $\{\underline{L}, L\} \to \{\underline{L}_a, L_a\}$ given by:

$$\underline{L}_a := a\underline{L}, \;\; L_a := \frac{1}{a}L$$

(for $a \in \mathcal{F}(\Sigma)$ a non-vanishing smooth function on Σ) gives $\langle \underline{L}_a, L_a\rangle = \langle \underline{L}, L\rangle = 2$ as well.

Our convention for the second fundamental form II and mean curvature $\vec{H}$ of Σ

are

$$\mathrm{II}(V,W) = D_V^{\perp} W, \quad \vec{H} = \mathrm{tr}_\Sigma \, \mathrm{II}$$

for $V, W \in \Gamma(T\Sigma)$ and D the Levi-Civita connection of the spacetime.

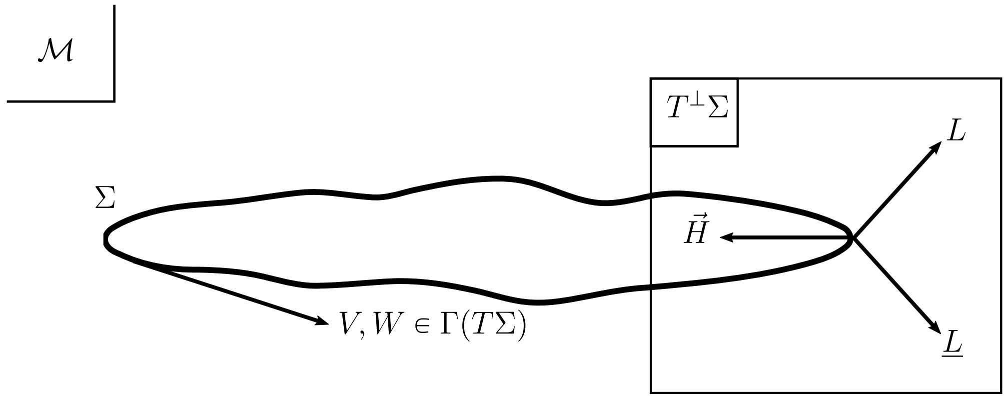

FIGURE 2.1: About a 2-sphere in spacetime

Definition 2.1.1. *Given a choice of null basis $\{\underline{L}, L\}$, following the conventions of Sauter [25], we define the associated symmetric 2-tensors $\underline{\chi}, \chi$ and torsion (connection 1-form) ζ by*

$$\underline{\chi}(V,W) := \langle D_V \underline{L}, W \rangle = -\langle \underline{L}, \mathrm{II}(V,W) \rangle$$

$$\chi(V,W) := \langle D_V L, W \rangle = -\langle L, \mathrm{II}(V,W) \rangle$$

$$\zeta(V) := \frac{1}{2}\langle D_V \underline{L}, L \rangle = -\frac{1}{2}\langle D_V L, \underline{L} \rangle$$

where $V, W \in \Gamma(T\Sigma)$.

Denoting the exterior derivative on Σ by $\not{d}$, any boosted basis $\{\underline{L}_a, L_a\}$ produces the associated tensors of Definition 2.1.1:

$$\underline{\chi}_a(V,W) := \langle D_V(a\underline{L}), W \rangle = a\underline{\chi}(V,W)$$

$$\chi_a(V,W) := \langle D_V(\frac{1}{a}L), W \rangle = \frac{1}{a}\chi(V,W)$$

$$\zeta_a(V) := \frac{1}{2}\langle D_V(a\underline{L}), \frac{1}{a}L \rangle = \zeta(V) + V \log|a| = (\zeta + \not{d} \log|a|)(V).$$

For a symmetric 2-tensor T on Σ its *trace-free* (or *trace-less*) part is given by

$$\hat{T} := T - \frac{1}{2}(\mathrm{tr}_\gamma T)\gamma$$

allowing us to decompose $\underline{\chi}$ into its *shear* and *expansion* components respectively:

$$\underline{\chi} = \hat{\underline{\chi}} + \frac{1}{2}(\mathrm{tr}\,\underline{\chi})\gamma.$$

Definition 2.1.2. *We say Σ is expanding along $\underline{L}$ for some null section $\underline{L} \in \Gamma(T^\perp\Sigma)$ provided that,*

$$\langle -\vec{H}, \underline{L}\rangle = \mathrm{tr}\,\underline{\chi} > 0 \qquad (\dagger)$$

on all of Σ.

Any infinitesimal flow of Σ along $\underline{L}$ gives, by first variation of area,

$$\dot{dA} = \langle -\vec{H}, \underline{L}\rangle dA = \mathrm{tr}\,\underline{\chi} dA.$$

So the flow is locally area expanding ($\dot{dA} > 0$) only if Σ "is expanding along $\underline{L}$":

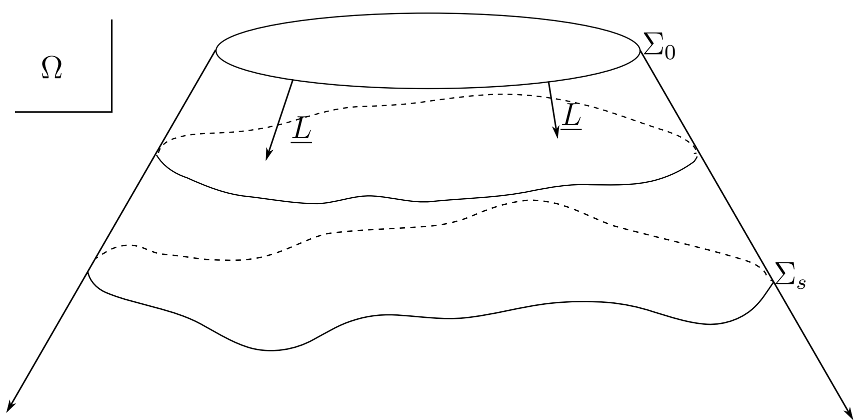

Figure 2.2: An expanding nullcone

Remark 2.1.1. *In Section 5.3 we will show (Lemma 5.3.1), whenever Ω is past asymptotically flat inside a spacetime satisfying the dominant energy condition, a* *consequence of the famous Raychaudhuri equation ((4.4), Section 4.2) is that any* $\Sigma \hookrightarrow \Omega$ *is expanding along* $\underline{L} \in \Gamma(T^{\perp}\Sigma) \cap \Gamma(T\Omega)|_{\Sigma}$ *the past pointing null section. So inequality (†) holds for any foliation of* Ω *along* $\underline{L}_a$ *where* $a > 0$ *and we have an expanding nullcone (as illustrated in the figure above).*

For Σ expanding along some $\underline{L} \in \Gamma(T^{\perp}\Sigma)$ we are able to choose a canonical null basis $\{L^-, L^+\}$ by requiring that our flow along $L^- = a\underline{L}$ be uniformly area expanding ($\dot{dA} = dA$). From first variation of area, flowing along $a\underline{L}$ gives

$$\dot{dA} = -\langle \vec{H}, a\underline{L} \rangle dA = a \operatorname{tr} \underline{\chi} dA.$$

So we achieve a uniformly area expanding null flow when $a = \frac{1}{\operatorname{tr}\underline{\chi}}$ giving:

Definition 2.1.3. *For* Σ *expanding along some* $\underline{L} \in \Gamma(T^{\perp}\Sigma)$ *we call the associated canonical uniformly area expanding null basis* $\{L^-, L^+\}$ *given by*

$$L^- := \frac{\underline{L}}{\operatorname{tr}\underline{\chi}}, \quad L^+ := \operatorname{tr}\underline{\chi} L$$

the null inflation basis.

We also define $\chi^{-(+)} := -\langle \mathrm{II}, L^{-(+)} \rangle$*. It follows from the comments proceeding Definition 2.1.1 that*

$$\operatorname{tr}\chi^- = 1$$

$$\operatorname{tr}\chi^+ = \operatorname{tr}\underline{\chi} \operatorname{tr}\chi = \langle \vec{H}, \vec{H} \rangle$$

and for $V \in \Gamma(T\Sigma)$ *the torsion associated to this basis is given by*

$$\tau(V) = \frac{1}{2}\langle D_V L^-, L^+ \rangle = (\zeta - \not{d} \log \operatorname{tr}\underline{\chi})(V).$$

We will denote the induced covariant derivative on Σ by $\not{\nabla}$.

Definition 2.1.4. *Assuming* Σ *is expanding along* $\underline{L}$*, for some* $\underline{L} \in \Gamma(T^{\perp}\Sigma)$*, we define the geometric flux function*

$$\rho = \mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H} \rangle + \nabla\!\!\!/ \cdot \tau \tag{2.1}$$

where $\mathcal{K}$ *represents the Gaussian curvature of* Σ*.*

This allows us to define the associated quasi-local mass

$$m(\Sigma) = \frac{1}{2}\Big(\frac{1}{4\pi}\int_{\Sigma} \rho^{\frac{2}{3}} dA\Big)^{\frac{3}{2}}. \tag{2.2}$$

For the induced covariant derivative $\nabla\!\!\!/$ we denote the associated Laplacian on Σ by $\Delta\!\!\!/$.

Remark 2.1.2. *Whenever* $\operatorname{tr}\chi^{+} = \langle \vec{H}, \vec{H} \rangle \neq 0$*,* Σ *has two null inflation bases given by* $\{L^{-}, L^{+}\}$ *and* $\{\frac{L^{+}}{\operatorname{tr}\chi^{+}}, \operatorname{tr}\chi^{+} L^{-}\}$*. As a result we typically have two distinct flux functions*

$$\rho_{-} = \mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H} \rangle + \nabla\!\!\!/ \cdot \tau$$

$$\rho_{+} = \mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H} \rangle - \nabla\!\!\!/ \cdot \tau - \Delta\!\!\!/ \log |\langle \vec{H}, \vec{H} \rangle|$$

with associated mass functionals $m_{\pm}$*. For the Bartnik datum* α_H *(see Definition 3.1.1), we will see for a past pointing* $\underline{L}$ *that* $\rho_{-} - \rho_{+} = 2\nabla\!\!\!/ \cdot \alpha_H$ *(Lemma 3.1.3). For* $\langle \vec{H}, \vec{H} \rangle \neq 0$*, whenever* Σ *is 'time-flat' (i.e.* $\nabla\!\!\!/ \cdot \alpha_H = 0$*) it follows that* $\rho_{-} = \rho_{+}$ *implies* $m_{-} = m_{+}$*.*

For a normal null flow off of some Σ with null flow vector $\underline{L}$, technically the flow speed is zero since $\langle \underline{L}, \underline{L} \rangle = 0$. In the case the Σ expands along $\underline{L}$ we define the expansion speed, σ, according to $\underline{L} = \sigma L^{-}$. We notice that $\sigma = \operatorname{tr}\underline{\chi}$. We are now ready to state our first result.

Theorem 2.1.1. *Let Ω be a null hypersurface foliated by spacelike spheres $\{\Sigma_s\}$ expanding along the null flow direction $\underline{L}$ such that $|\rho(s)| > 0$ for each s. Then the mass $m(s) := m(\Sigma_s)$ has rate of change*

$$\frac{dm}{ds} = \frac{(2m)^{\frac{1}{3}}}{8\pi}\int_{\Sigma_s}\frac{\sigma}{\rho^{\frac{1}{3}}}\Big((|\hat{\chi}^-|^2+G(L^-,L^-))(\frac{1}{4}\langle\vec{H},\vec{H}\rangle-\not{\Delta}\log|\rho|^{\frac{1}{3}})+\frac{1}{2}|\eta_\rho|^2+G(L^-,N)\Big)dA$$

where

- *G is the Einstein tensor for the ambient metric g*
- *$\sigma = \mathrm{tr}\,\underline{\chi}$ is the expansion speed*
- *$\eta_\rho := 2\hat{\chi}^- \cdot \not{d}\log|\rho|^{\frac{1}{3}} - \tau$*
- *$N := |\not{d}\log|\rho|^{\frac{1}{3}}|^2 L^- + \not{\nabla}\log|\rho|^{\frac{1}{3}} - \frac{1}{4}L^+$*

If we assume therefore that our spacetime M satisfies the dominant energy condition we can show our mass functional $m(\Sigma_s)$ is non-decreasing for foliations $\{\Sigma_s\}$ satisfying the following convexity condition:

Definition 2.1.5. *Given a foliation of 2-spheres $\{\Sigma_s\}_{s\geqslant 0}$ we say it is a (P)-foliation provided:*

$$\rho > 0$$
$$\frac{1}{4}\langle\vec{H},\vec{H}\rangle \geqslant \not{\Delta}\log\rho^{\frac{1}{3}}$$

is satisfied on each Σ_s. We say $\{\Sigma_s\}_{s\geqslant 0}$ is a strict (P)-foliation or (SP)-foliation if additionally:

$$\frac{1}{4}\langle\vec{H},\vec{H}\rangle = \not{\Delta}\log\rho^{\frac{1}{3}},\ \textit{for } s = 0$$
$$\frac{1}{4}\langle\vec{H},\vec{H}\rangle > \not{\Delta}\log\rho^{\frac{1}{3}},\ \textit{for } s > 0.$$

So for a (P)-foliation the dominant energy condition ensures the product of the first two terms of the integrand in Theorem 2.1.1 be non-negative. The second is non-negative since each Σ_s is spacelike and the last term is non-negative from the dominant energy condition since $\langle N, N\rangle = 0$ and $\langle N, L^-\rangle = -\frac{1}{2} < 0$ (i.e. N is null and at every point $p \in \Sigma$ lies inside the same connected component of the nullcone in $T_p\mathcal{M}$ as L^-).

We will assume in Chapters 5 and 6 that $\underline{L}$ is past pointing. Adopting the same definitions as Mars and Soria [18] (see Section 5.3) we have our second main result:

Theorem 2.1.2. *Let Ω be a null hypersurface in a spacetime satisfying the dominant energy condition that extends to past null infinity. Then given the existence of a (P)-foliation $\{\Sigma_s\}$ we have*

$$m(0) \leqslant \lim_{s\to\infty} m(\Sigma_s) =: M$$

(for $M \leqslant \infty$). If, in addition, Ω is past asymptotically flat with strong flux decay and $\{\Sigma_s\}$ asymptotically geodesic (see Section 5.3) then

$$M \leqslant m_B$$

where m_B is the Bondi mass of Ω. Moreover, in the case that $\operatorname{tr}\chi|_{\Sigma_0} = 0$ we have the null Penrose inequality

$$\sqrt{\frac{|\Sigma_0|}{16\pi}} \leqslant m_B.$$

Furthermore, when equality holds for an (SP)-foliation we conclude that equality holds for all foliations of Ω and the data (γ, $\underline{\chi}$, $\operatorname{tr}\chi$ and ζ) agree with some foliation of the standard nullcone of the Schwarzschild spacetime.

2.2 Final introductory remarks

Recalling the Hawking Energy for a closed spacelike surface Σ

$$E_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\Big(1 - \frac{1}{16\pi}\int_\Sigma \langle \vec{H}, \vec{H}\rangle dA\Big)$$

we notice by the Gauss-Bonnet and Divergence Theorems that

$$\int_\Sigma \rho dA = 8\pi \frac{E_H(\Sigma)}{\sqrt{\frac{|\Sigma|}{4\pi}}}$$

motivating in part why we call ρ a flux function.

As mentioned previously, one such flux introduced by Christodoulou [9], is the 'mass aspect function'

$$\mu = \mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H}\rangle - \nabla\!\!\!/ \cdot \zeta$$

associated to an arbitrary null basis $\{\underline{L}, L\} \subset \Gamma(T^\perp\Sigma)$. Using μ in his Ph.D thesis [25], Sauter showed the existence of flows on past nullcones that render E_H non-decreasing making explicit use of the fact that under a boost this mass aspect function changes via ζ according to

$$\zeta \to \zeta_a = \zeta + d\!\!\!/ \log|a| \implies \mu \to \mu_a = \mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H}\rangle - \nabla\!\!\!/ \cdot \zeta - \Delta\!\!\!/ \log|a|.$$

From these observations, the divergence term in (2.1) (up to a sign) is somewhat motivated by an attempt to find a flux function independent of boosts. In fact, it follows in the case that $0 < \langle \vec{H}, \vec{H}\rangle =: H^2$ and $\underline{L}$ is past pointing, that ρ can be given in terms of the Bartnik data of Σ as

$$\rho = \mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H}\rangle + \nabla\!\!\!/ \cdot \alpha_H - \Delta\!\!\!/ \log H$$

(we refer the reader to Chapter 3 for definitions and proof). Moreover, in our two simplest cases, namely spherical cross-sections of the nullcone of a point in a space form or the standard nullcone of Schwarzchild spacetime, the last two terms cancel identically. Interestingly, work of Wang, Wang and Zhang [30] show deep connection between the 1-form $\alpha_H - \not{d} \log H$ and the underlying null geometry of a closed, co-dimension 2 surface Σ. For Σ satisfying $\alpha_H = \not{d} \log H$ they show for various ambient structures that Σ must be constrained to a shear-free ($\hat{\underline{\chi}} = 0$) null-hypersurface of spherical symmetry. In Chapter 3 (Proposition 3.1.1) we show for a connected Σ of arbitrary co-dimension inside a space form, if Σ is expanding along some null section $\underline{L} \in \Gamma(T^{\perp}\Sigma)$ such that $D^{\perp}\underline{L} \propto \underline{L}$, then it must be constrained to the nullcone of point whenever $\hat{\underline{\chi}} = 0$. Leaning on work by Bray, Jauregui and Mars ([7]) we also find direct motivation for (2.1) showing that ρ arises naturally from variation of E_H along null flows.

3

Motivation

In this chapter we further develop our motivation for ρ based on an analysis of nullcones within space forms. We then provide analysis of standard nullcones in Kruskal spacetime to motivate $m(\Sigma)$ and for comparison with (SP)-foliations satisfying $\frac{dm}{ds} = 0$. We also show how an arbitrary variation of E_H building on work of Bray, Jauregui and Mars [7], points toward ρ being the optimal choice of flux function in the case of null flows. In this thesis we will be using the following convention to construct the Riemann curvature tensor:

$$R_{XY}Z := D_{[X,Y]}Z - [D_X, D_Y]Z.$$

From this will have need of the following versions of the Gauss and Codazzi equations:

Proposition 3.0.1. *Suppose Σ is a co-dimension 2 semi-Riemannian submanifold of $\mathcal{M}^{n+1}$ that locally admits a normal null basis $\{\underline{L}, L\}$ such that $\langle \underline{L}, L\rangle = 2$. Then,*

$$(n-1)\mathcal{K} - \frac{n-2}{n-1}\langle \vec{H}, \vec{H}\rangle + \underline{\hat{\chi}} \cdot \hat{\chi} = -R - 2G(\underline{L}, L) - \frac{1}{2}\langle R_{\underline{L}L}\underline{L}, L\rangle \tag{3.1}$$

$$\nabla\!\!\!/ \cdot \underline{\hat{\chi}}(V) - \underline{\hat{\chi}}(V, \vec{\zeta}) + \frac{n-2}{n-1}\operatorname{tr}\underline{\chi}\zeta(V) - \frac{n-2}{n-1}V tr\underline{\chi} = G(V, \underline{L}) - \frac{1}{2}\langle R_{\underline{L}V}L, \underline{L}\rangle \tag{3.2}$$

for $V \in \Gamma(T\Sigma)$ and $(n-1)\mathcal{K}$ the scalar curvature of Σ.

Proof. From the Gauss equation ([21],pg.100) we have,

$$\langle \not{R}_{V,W}U,S\rangle = \langle R_{V,W}U,S\rangle + \langle \vec{\mathrm{II}}(V,U),\vec{\mathrm{II}}(W,S)\rangle - \langle \vec{\mathrm{II}}(V,S),\vec{\mathrm{II}}(W,U)\rangle$$

for $\not{R}$, R the Riemann tensors of $\Sigma,\mathcal{M}$ respectively and $V,W,U,S \in \Gamma(T\Sigma)$. Restricted to Σ the ambient metric has inverse

$$g^{-1}|_\Sigma = \frac{1}{2}\underline{L}\otimes L + \frac{1}{2}L\otimes\underline{L} + \gamma^{-1}$$

so taking a trace over V,U then W,S in Σ we have:

$$\langle \not{R}_{VW}U,S\rangle \xrightarrow{(V,U)} \not{R}ic(W,S) \xrightarrow{(W,S)} (n-1)\mathcal{K}$$

$$\langle R_{VW}U,S\rangle \xrightarrow{(V,U)} Ric(W,S) - \frac{1}{2}(\langle R_{\underline{L}W}L,S\rangle + \langle R_{LW}\underline{L},S\rangle)$$

$$\xrightarrow{(W,S)} R - 2Ric(\underline{L},L) - \frac{1}{2}\langle R_{\underline{L}L}\underline{L},L\rangle.$$

Since $\vec{\mathrm{II}} = \hat{\mathrm{II}} + \frac{1}{n-1}\vec{H}\gamma = -\frac{1}{2}\underline{\hat{\chi}}L - \frac{1}{2}\hat{\chi}\underline{L} + \frac{1}{n-1}\vec{H}\gamma$ we have

$$\langle\vec{\mathrm{II}},\vec{\mathrm{II}}\rangle = \frac{1}{2}(\underline{\hat{\chi}}\otimes\hat{\chi} + \hat{\chi}\otimes\underline{\hat{\chi}}) - \frac{\langle\vec{H},L\rangle}{2(n-1)}(\underline{\hat{\chi}}\otimes\gamma + \gamma\otimes\underline{\hat{\chi}}) - \frac{\langle\vec{H},\underline{L}\rangle}{2(n-1)}(\hat{\chi}\otimes\gamma + \gamma\otimes\hat{\chi})$$

$$+ \left(\frac{1}{n-1}\right)^2\langle\vec{H},\vec{H}\rangle\gamma\otimes\gamma$$

so returning to our trace

$$\langle\vec{\mathrm{II}}(V,U),\vec{\mathrm{II}}(W,S)\rangle \xrightarrow{(V,U),(W,S)} \langle\vec{H},\vec{H}\rangle$$

$$\langle\vec{\mathrm{II}}(V,S),\vec{\mathrm{II}}(W,U)\rangle \xrightarrow{(V,U),(W,S)} \underline{\hat{\chi}}\cdot\hat{\chi} + \frac{1}{n-1}\langle\vec{H},\vec{H}\rangle.$$

Equating terms according to the Gauss equation we have

$$(n-1)\mathcal{K} = R - 2Ric(\underline{L},L) - \frac{1}{2}\langle R_{\underline{L}L}\underline{L},L\rangle - \underline{\hat{\chi}}\cdot\hat{\chi} + (1-\frac{1}{n-1})\langle\vec{H},\vec{H}\rangle$$

$$= -R - 2G(\underline{L},L) - \frac{1}{2}\langle R_{\underline{L}L}\underline{L},L\rangle - \underline{\hat{\chi}}\cdot\hat{\chi} + \frac{n-2}{n-1}\langle\vec{H},\vec{H}\rangle$$

having used $G(\cdot,\cdot) = Ric(\cdot,\cdot) - \frac{1}{2}R\langle\cdot,\cdot\rangle$, (3.1) follows.

From the Codazzi equation ([21],pg.115), for any $V, W, U \in \Gamma(T\Sigma)$,

$$R^{\perp}_{VW}U = -(\nabla\!\!\!/_V \mathrm{II})(W,U) + (\nabla\!\!\!/_W \mathrm{II})(V,U)$$

where

$$(\nabla\!\!\!/_V \mathrm{II})(W,U) := D^{\perp}_V(\mathrm{II}(W,U)) - \mathrm{II}(\nabla\!\!\!/_V W, U) - \mathrm{II}(W, \nabla\!\!\!/_V U).$$

So given our choice of null normal $\underline{L}$ we see that

$$\begin{aligned}
\langle D^{\perp}_V(\mathrm{II}(W,U)), \underline{L}\rangle &= -V(\underline{\chi}(W,U)) - \langle \mathrm{II}(W,U), D_V\underline{L}\rangle \\
&= -V(\underline{\chi}(W,U)) - \frac{1}{2}\langle \mathrm{II}(W,U), \underline{L}\rangle\langle L, D_V\underline{L}\rangle \\
&= -V(\underline{\chi}(W,U)) + \underline{\chi}(W,U)\zeta(V) \\
\langle (\nabla\!\!\!/_V \mathrm{II})(W,U), \underline{L}\rangle &= -V(\underline{\chi}(W,U)) + \underline{\chi}(W,U)\zeta(V) + \underline{\chi}(\nabla\!\!\!/_V W, U) + \underline{\chi}(W, \nabla\!\!\!/_V U) \\
&= \zeta(V)\underline{\chi}(W,U) - (\nabla\!\!\!/_V \underline{\chi})(W,U).
\end{aligned}$$

Therefore,

$$\langle R_{VW}U, \underline{L}\rangle = (\nabla\!\!\!/_V \underline{\chi})(W,U) - (\nabla\!\!\!/_W \underline{\chi})(V,U) - \zeta(V)\underline{\chi}(W,U) + \zeta(W)\underline{\chi}(V,U).$$

Taking a trace over V, U we conclude,

$$\begin{aligned}
Ric(W,\underline{L}) - \frac{1}{2}\langle R_{\underline{L},W}L, \underline{L}\rangle &= \nabla\!\!\!/ \cdot \underline{\chi}(W) - W tr\underline{\chi} - \underline{\chi}(W,\vec{\zeta}) + tr\underline{\chi}\zeta(W) \\
&= \nabla\!\!\!/ \cdot \hat{\underline{\chi}}(W) - \frac{n-2}{n-1} W tr\underline{\chi} - \hat{\underline{\chi}}(W,\vec{\zeta}) + \frac{n-2}{n-1} tr\underline{\chi}\zeta(W)
\end{aligned}$$

and notice that $G(W,\underline{L}) = Ric(W,\underline{L})$ since $\langle \underline{L}, W\rangle = 0$. □

3.1 Nullcone of a point in a Space Form

In this section we spend some time studying ρ and $m(\Sigma)$ on cross-sections of the nullcone of a point in a space form. We adopt the notation as in [21] where $\mathbb{R}^n_\nu$ corresponds to the manifold $\mathbb{R}^n$ endowed with the standard inner product of index ν.

Lemma 3.1.1. *Suppose $\Sigma^k \hookrightarrow \mathbb{R}^n_\nu$ ($k \geqslant 2$) is a connected semi-Riemannian submanifold admitting a non-trivial section $\vec{n} \in \Gamma(T^\perp\Sigma)$ such that $D^\perp \vec{n} = \eta\vec{n}$ for some 1-form η. Then the following are equivalent*

1. *$p \mapsto \exp(-\vec{n}|_p)$ is constant*

2. *$\eta = 0$ and $\langle \mathrm{II}, -\vec{n}\rangle = \gamma$*

3. *$\langle \mathrm{II}, -\vec{n}\rangle = \gamma$*

where $\gamma = \langle \cdot, \cdot \rangle|_\Sigma$ and $\exp : T\mathbb{R}^n_\nu \to \mathbb{R}^n_\nu$ is the exponential map.

Proof. Choosing an origin $\vec{o}$ for $\mathbb{R}^n_\nu$ with associated position vector field given by $P = x^i\partial_i \in \Gamma(T\mathbb{R}^n_\nu)$ it follows that

$$\exp(-\vec{n}|_{\vec{p}}) = (P - \vec{n})|_{\vec{p}}$$

where, by an abuse of notation, we have omitted the composition of canonical isometries $T_{\vec{p}}\mathbb{R}^n_\nu \to T_{\vec{o}}\mathbb{R}^n_\nu \to \mathbb{R}^n_\nu$ identifying $\vec{p}$ with $P|_{\vec{p}}$. As a result, for any $V \in \Gamma(T\Sigma)$:

$$\begin{aligned} d(\exp(-\vec{n}))(V) &= D_V(P - \vec{n}) \\ &= V - D_V\vec{n} \\ &= (V - D^\parallel_V\vec{n}) - \eta(V)\vec{n} \end{aligned}$$

and we conclude that $\exp\circ(-\vec{n})$ is locally constant (or constant when Σ is connected) if and only if both $V = D^\parallel_V\vec{n}$ for any $V \in \Gamma(T\Sigma)$ and $\eta = 0$. Since $D^\parallel_V\vec{n} = V$ for any $V \in \Gamma(T\Sigma)$ is equivalent to $-\langle \mathrm{II}(V,W), \vec{n}\rangle (= \langle W, D_V\vec{n}\rangle) = \langle V, W\rangle$ for any $V, W \in \Gamma(T\Sigma)$ we have that 1. $\Longleftrightarrow$ 2.

2. $\Longrightarrow$ 3. is trivial. To show 3. $\Longrightarrow$ 2. we start by taking any $U, V, W \in \Gamma(T\Sigma)$ so that the Codazzi equation gives

$$\langle (\bar{\nabla}_V \mathrm{II})(W, U), \vec{n}\rangle = \langle (\bar{\nabla}_W \mathrm{II})(V, U), \vec{n}\rangle$$

where

$$\begin{aligned}\langle(\nabla\!\!\!/_V \mathrm{II})(W,U),\vec{n}\rangle &:= \langle D_V^{\perp}(\mathrm{II}(W,U)) - \mathrm{II}(\nabla\!\!\!/_V W,U) - \mathrm{II}(W,\nabla\!\!\!/_V U),\vec{n}\rangle \\ &= -(\nabla\!\!\!/_V \gamma)(W,U) - \langle \mathrm{II}(W,U), D_V\vec{n}\rangle \\ &= \eta(V)\langle W,U\rangle\end{aligned}$$

and therefore $\eta(V)\langle W,U\rangle = \eta(W)\langle V,U\rangle$. Taking a trace over V,U we conclude that $\eta(W) = k\eta(W)$ so that $k \geqslant 2$ forces $\eta = 0$ as desired. □

The Hyperquadrics of $\mathbb{R}^n_\nu$ correspond to the complete, totally umbilic hypersurfaces H_C of constant curvature C (provided $C \neq 0$) given by

$$H_C := \{\vec{v} \in \mathbb{R}^n_\nu | \langle\vec{v},\vec{v}\rangle = C\}$$

where C runs through all values in $\mathbb{R}$. When $C = 0$, $\Omega = H_0$ is the collection of all null geodesics emanating from the origin called the *nullcone* centered at the origin. Consequently $\Omega + \vec{p}$ corresponds to the nullcone at the point $\vec{p}$. Similarly for a space form $\mathcal{M}$ we will define the nullcone of a point $p \in \mathcal{M}$ as the collection of all null geodesics emanating from p.

Proposition 3.1.1. *Suppose $\Sigma^k \hookrightarrow \mathcal{M}^{n-1}$ $(k \geqslant 2)$ is a connected semi-Riemannian submanifold of a space form $\mathcal{M}$. Suppose that Σ is expanding along some null section $\underline{L}$ satisfying $D^{\perp}\underline{L} = \zeta\underline{L}$ for some 1-form (or torsion) ζ. Then the following are equivalent*

1. $p \mapsto \exp(-\frac{k\underline{L}}{\mathrm{tr}\,\underline{\chi}}|_p)$ *is constant*

2. $\tau := \zeta - d\!\!\!/ \log \mathrm{tr}\,\underline{\chi} = 0$ *and* $\hat{\underline{\chi}} = 0$

3. $\hat{\underline{\chi}} = 0$

where $\underline{\chi} := -\langle\underline{L}, \mathrm{II}\rangle$.

Proof. If $\mathcal{M}$ has constant curvature $C \neq 0$ we find a Hyperquadric H_C of $\mathbb{R}^n_\nu$ (for some ν) of the same dimension and index as $\mathcal{M}$. It's a well known fact that $\mathcal{M}$ and H_C have isometric semi-Riemannian coverings ([21],pg.224 Theorem 17) which we identify and denote by $\mathcal{O}$. As a result, for any $q \in \Sigma \subset \mathcal{M}$ we find a $\vec{q} \in H_C$ with isometric neighborhoods. Moreover, we find an open set $U_q \subset \Sigma$ of q which is isometric onto some $V_{\vec{q}} \subset H_C$. Without loss of generality we will also identify $T^\perp(U_q)$ and $T^\perp(V_{\vec{q}})$. Denoting the ambient connection on $\mathbb{R}^n_\nu$ by $\bar{D}$ and the unit normal of $H^{n-1}_C \subset \mathbb{R}^n_\nu$ by $\vec{N}$ we conclude, for the null section $L^- = \frac{L}{\operatorname{tr}\underline{\chi}}$, with the decomposition $\bar{D}^\perp_V L^- = \tau(V)L^- + \langle \vec{N}, \vec{N} \rangle \langle \bar{D}_V L^-, \vec{N} \rangle \vec{N}$. So given that all Hyperquadrics are totally umbilic it follows that $\langle \bar{D}_V L^-, \vec{N} \rangle \propto \langle V, L^- \rangle = 0$ and therefore $\bar{D}^\perp_V L^- = \tau(V)L^-$.

We wish to show 3. $\implies$ 1. From the hypothesis we have that $\underline{\chi} = \frac{1}{k} \operatorname{tr}\underline{\chi}\gamma$ so it follows that $k\underline{\chi}^- = -\langle kL^-, \mathrm{II} \rangle = \gamma$ and Lemma 3.1.1 applies for $V_{\vec{q}} \subset \mathbb{R}^n_\nu$. We conclude that $V_{\vec{q}}$ is contained inside the nullcone of a point $\vec{o} \in \mathbb{R}^n_\nu$, where $\vec{o} = \exp_{n,\nu}(-kL^-)|_{V_{\vec{q}}}$, and every $\vec{p} \in V_{\vec{q}}$ is connected to $\vec{o}$ by a null geodesic in $\mathbb{R}^n_\nu$ along $kL^-|_{\vec{p}} = \left.\frac{kL}{\operatorname{tr}\underline{\chi}}\right|_{\vec{p}} \in T^\perp_{\vec{p}} V_{\vec{q}} \subset T_{\vec{p}} H_C$. Since H_C is complete and totally umbilic these null geodesics must remain within H_C. Up to a possible shrinkage of $V_{\vec{q}}$ we may lift a neighborhood of the geodesic $\vec{q} \to \vec{o}$ to a neighborhood of some null geodesic $\tilde{q} \to \tilde{o}$ in $\mathcal{O}$ concluding that the isometric image $V_{\tilde{q}}$ of $V_{\vec{q}}$ contracts to $\tilde{o}$ along null geodesics. Since $\mathcal{M}$ is complete the null geodesic $\tilde{q} \to \tilde{o}$ in turn gives rise to a null geodesic $q \to o$ in $\mathcal{M}$ and up to an additional shrinkage we conclude that U_q contracts along null geodesics onto o:

In fact our argument shows that the union of all points in Σ that get transported to o must form an open subset of Σ. Conversely, if any point in Σ gets transported to a point other than o the same follows for a neighborhood around that point in Σ. By connectedness, all of Σ must be transported to o along null geodesics as desired.

For 1. $\implies$ 3. we take a null geodesic from $q \in \Sigma$ along $kL^- = \frac{kL}{\operatorname{tr}\underline{\chi}}$ to the focal

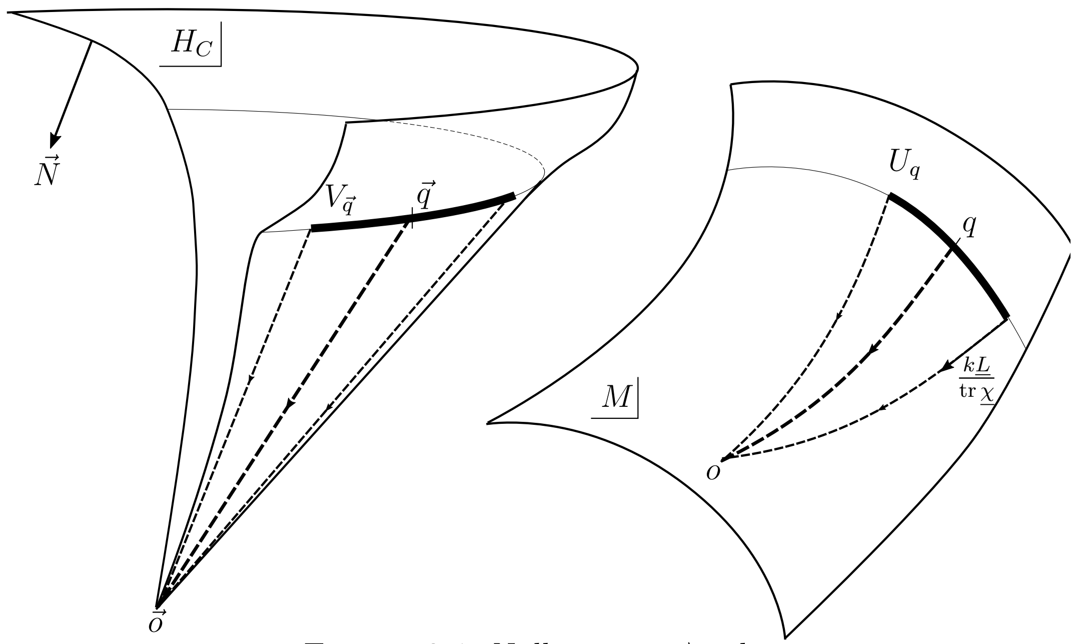

FIGURE 3.1: Nullcones at $\vec{o}$ and o

point, at say, $o \in \mathcal{M}$. Similarly as before this gives rise to a tubular neighborhood around some null geodesic $\vec{q} \to \vec{o}$ in H_C within which $V_{\vec{q}}$ is contracted along null geodesics onto $\vec{o}$. Since H_C is totally umbilic $V_{\vec{q}}$ is transported to $\vec{o}$ along null geodesics in $\mathbb{R}^n_\nu$ forming part of the nullcone at $\vec{o} = \exp_{n,\nu}(-kL^-)|_{V_{\vec{q}}}$. Lemma 3.1.1 applies once again and we conclude that $k\chi^- = \gamma$ implies $\hat{\underline{\chi}} = 0$ on $V_{\vec{q}}$ hence on U_q (since they have isometric neighborhoods). Since q was arbitrary chosen the result follows.

Once again 2. $\implies$ 3. is trivial. To show 3. $\implies$ 2. we have similarly as in Lemma 3.1.2 from the Codazzi equation for $\Sigma \hookrightarrow \mathcal{M}$ and $\mathcal{M}$ of constant curvature that:

$$\tau(V)\langle W, U\rangle = \tau(W)\langle V, U\rangle$$

so that a trace over V, U yields again $\tau(W) = k\tau(W)$ and therefore $\tau = 0$. □

For any connected, co-dimension 2 surface admitting a null section $\underline{L} \in \Gamma(T^\perp\Sigma)$, since $\langle D_V \underline{L}, \underline{L}\rangle = \frac{1}{2}V\langle \underline{L}, \underline{L}\rangle = 0$, it necessarily follows that $D^\perp \underline{L} = \zeta \underline{L}$ for some associated 1-form ζ. In particular, Σ will be contained inside the nullcone of a point inside a space form $\mathcal{M}$ if we're able to find a null section $\underline{L}$ along which Σ is expanding and shear-free. Along such $\underline{L}$ it follows from Lemma 3.1.1 for $C = 0$ and Proposition 3.1.1 for $C \neq 0$ that $\tau = 0$. So for Lorentzian space forms of dimension-4 (i.e. 'Minkowski spacetime' for $C = 0$, 'de Sitter spacetime' for $C > 0$ and 'anti-de Sitter spacetime' for $C < 0$) it follows from (3.1) that Σ has flux $\rho = K - \frac{1}{4}\langle \vec{H}, \vec{H} \rangle = C$. When Σ is a 2-sphere, by the Gauss-Bonnet Theorem, we conclude that

$$m(\Sigma) = |E_H(\Sigma)| = \frac{|C|}{2}\left(\frac{|\Sigma|}{4\pi}\right)^{\frac{3}{2}}.$$

The reader may be wondering why the need for the divergence term in (2.1) when it vanishes altogether. We take as our first hint the fact that vanishing $\tau = \zeta - \not{d} \log \operatorname{tr} \underline{\chi}$ is characteristic of spherical cross-sections of Ω which subsequently may obscure it's contribution. In the paper by Wang, Wang and Zhang ([30] Theorem 3.13, Theorem 5.2) the authors prove $\tau = 0$ to be sufficient in spacetimes of constant curvature to constrain a closed, co-dimension 2 surface Σ to a shear-free null hypersurface of spherical symmetry. Proof follows from the following Lemma and Proposition 3.1.1 when Σ is a 2-sphere:

Lemma 3.1.2. *Suppose Σ is a spacelike 2-sphere expanding along some $\underline{L}$ inside a space form $\mathcal{M}$. Suppose also $D^{\perp}\underline{L} = \zeta \underline{L}$ for some 1-form ζ then*

$$\tau := \zeta - \not{d} \log \operatorname{tr} \underline{\chi} = 0 \implies \hat{\underline{\chi}} = 0.$$

Proof. As used in Proposition 3.1.1 to prove the implication in the opposite direction, we start with the Codazzi equation. For $L^- = \frac{\underline{L}}{\operatorname{tr} \underline{\chi}}$ we recall that $D^{\perp}L^- = \tau L^- = 0$ and $\operatorname{tr} \chi^- = 1$ so we have:

$$\begin{aligned} \langle \nabla\!\!\!/_V \operatorname{II}(W, U), L^- \rangle &= \langle D_V^{\perp}(\operatorname{II}(W, U)) - \operatorname{II}(\nabla\!\!\!/_V W, U) - \operatorname{II}(W, \nabla\!\!\!/_V U), L^- \rangle \\ &= -(\nabla\!\!\!/_V \chi^-)(W, U) - \langle \operatorname{II}(W, U), D_V^{\perp} L^- \rangle \\ &= -(\nabla\!\!\!/_V \chi^-)(W, U) \end{aligned}$$

$$= -(\nabla\!\!\!/_V \hat{\chi}^-)(W,U)$$

$$0 = \langle R^{\perp}_{VW} U, L^- \rangle = \langle -\nabla\!\!\!/_V \, \mathrm{II}(W,U) + \nabla\!\!\!/_W \, \mathrm{II}(V,U), L^- \rangle$$

$$= (\nabla\!\!\!/_V \hat{\chi}^-)(W,U) - (\nabla\!\!\!/_W \hat{\chi}^-)(V,U).$$

Taking a trace over V, U this implies $\nabla\!\!\!/ \cdot \hat{\chi}^- = 0$. Since Σ is a topological 2-sphere it's a well known consequence of the Uniformization Theorem (see for example [9]) that the divergence operator on symmetric trace-free 2-tensors is injective so that $\underline{\hat{\chi}} = \mathrm{tr}\,\underline{\chi}\hat{\chi}^- = 0$. □

Definition 3.1.1. *We say a 2-sphere Σ is* *<u>admissible</u>* *if*

$$\langle \vec{H}, \vec{H} \rangle = H^2 > 0.$$

In the case that Σ is admissible we're able to construct the orthonormal frame field

$$\{e_r = -\frac{\vec{H}}{H}, e_t\}$$

for e_t future pointing. The associated connection 1-form is given by

$$\alpha_H(V) := \langle D_V e_r, e_t \rangle.$$

From the following known Lemma ([30]), Proposition 3.1.1 and Lemma 3.1.2, a necessary and sufficient condition for an admissible sphere Σ to be constrained to the past(future) light-cone of a point in a space form is given by $\alpha_H = \pm d\!\!\!/ \log H$:

Lemma 3.1.3. *For Σ admissible*

$$\tau = \pm\alpha_H - d\!\!\!/ \log H$$

from which we conclude that

$$\rho_{\mp} = K - \frac{1}{4}\langle \vec{H}, \vec{H} \rangle \pm \nabla\!\!\!/ \cdot \alpha_H - \Delta\!\!\!\!/ \log H$$

where $+/-$ indicates whether L^- is past/future pointing.

Proof. Since $-\vec{H} = \frac{1}{2}(\operatorname{tr}\chi \underline{L} + \operatorname{tr}\underline{\chi} L)$ we see $H^2 = \operatorname{tr}\underline{\chi}\operatorname{tr}\chi$ so that the inverse mean curvature vector is given by

$$\vec{I} := -\frac{\vec{H}}{H^2} = \frac{1}{2}\Big(\frac{\underline{L}}{\operatorname{tr}\underline{\chi}} + \frac{L}{\operatorname{tr}\chi}\Big).$$

As a result,

$$\begin{aligned}
\alpha_H(V) &= \langle D_V e_r, e_t\rangle \\
&= \langle D_V \frac{e_r}{H}, He_t\rangle \\
&= \langle D_V \frac{1}{2}\Big(\frac{\underline{L}}{\operatorname{tr}\underline{\chi}} + \frac{L}{\operatorname{tr}\chi}\Big), \mp\frac{1}{2}(\operatorname{tr}\chi\underline{L} - \operatorname{tr}\underline{\chi}L)\rangle \\
&= \pm\frac{1}{4}\Big(\langle D_V\frac{\underline{L}}{\operatorname{tr}\underline{\chi}}, \operatorname{tr}\underline{\chi}L\rangle - \langle D_V\frac{\operatorname{tr}\underline{\chi}L}{H^2}, H^2\frac{\underline{L}}{\operatorname{tr}\underline{\chi}}\rangle\Big) \\
&= \pm\frac{1}{4}\Big(\langle D_V\frac{\underline{L}}{\operatorname{tr}\underline{\chi}}, \operatorname{tr}\underline{\chi}L\rangle + \langle D_V(H^2\frac{\underline{L}}{\operatorname{tr}\underline{\chi}}), \frac{\operatorname{tr}\underline{\chi}L}{H^2}\rangle\Big) \\
&= \pm\frac{1}{4}\Big(2\langle D_V\frac{\underline{L}}{\operatorname{tr}\underline{\chi}}, \operatorname{tr}\underline{\chi}L\rangle + 2V\log H^2\Big) \\
&= \pm\Big(\zeta(V) - V\log\operatorname{tr}\underline{\chi} + V\log H\Big)
\end{aligned}$$

□

Wang, Wang and Zhang ([30] Theorem B') also extend their result to expanding, co-dimension 2 surfaces Σ in n-dimensional Schwarzschild spacetime ($n \geqslant 4$). Namely, that any such Σ satisfying $\alpha_H = d\log H$ must be inside a shear-free null hypersurfaces of symmetry, or the 'standard nullcone' in this geometry. So with the hopes of further illuminating modification of E_H by way of the flux function ρ we move on to this setting in dimension 4.

3.2 Standard Nullcones of Schwarzschild Revisited

Recall the Kruskal spacetime $\mathbb{P} \times_r \mathbb{S}^2$ with associated quadratic form:

$$ds^2 = 2F(r)dvdu + r^2(d\vartheta^2 + (\sin\vartheta)^2 d\varphi^2).$$

A standard past nullcone of Schwarzchild spacetime Ω is the hypersurface given by fixing the coordinate at $v = v_0 > 0$. Denoting the gradient of a function f by Df we recognize the null vector field $\frac{\partial_u}{F} = Dv$ restricts to Ω as both a tangent (since $\partial_u(v) = 0$) and normal (since $Dv \perp T\Omega$) vector field. It follows that $Dv \in T^\perp\Omega \cap T\Omega$ and the induced metric on Ω degenerate, so Ω is an example of a *null hypersurface*. From the identity $D_{Df}Df = \frac{1}{2}D|Df|^2$ we see $\frac{\partial_u}{F}$ is geodesic and Ω is realized as the past light cone of a section of the event horizon at $r = 2M$ (see Figure 1.4).
Setting $\underline{L} = D(4M \log v) = \frac{4M}{v}\frac{\partial_u}{F}$ we see $\underline{L}(r) = \frac{4M}{v}\frac{r_u}{F} = \frac{4M}{v}\frac{v}{g'(r)F} = \frac{4M}{v}\frac{v}{4M} = 1$. We conclude that r restricts to an affine parameter along the geodesics generating Ω and therefore any cross-section Σ can be given as a graph over $\mathbb{S}^2$ in Ω with graph function $\omega = r|_\Sigma$. We extend ω to the rest of Ω by assigning $\underline{L}(\omega) = 0$ and to a neighborhood of Ω by assigning $\partial_v\omega = 0$. From the canonical, homothetic embedding onto the leaves $\mathbb{S}^2 \hookrightarrow \mathbb{P} \times_r \mathbb{S}^2$ we obtain the lifted vector fields $V \in \mathcal{L}(\mathbb{S}^2) \subset \Gamma(T(\mathbb{P} \times_r \mathbb{S}^2))$ such that $\langle \partial_u(\partial_v), V\rangle = [\partial_v(\partial_u), V] = 0$. It follows that $\mathcal{L}(\mathbb{S}^2)|_{\Sigma_r} = \Gamma(T\Sigma_r)$ (for $\Sigma_r := \{r = const., v = v_0\}$) and therefore $\tilde{V} := V + V\omega\underline{L} \in \Gamma(T\Sigma)$ since

$$\tilde{V}(r - \omega) = -V\omega + V\omega\underline{L}(r) = -V\omega + V\omega = 0.$$

Since $\Sigma = \{4M \log \frac{v}{v_0} = 0, r = \omega\}$ we have $\underline{L}, D(r - \omega) \in \Gamma(T^\perp\Sigma)$ are linearly independent so that $L = a\underline{L} + bD(r - \omega)$ and we wish to solve for a, b. Also, we have $Dr = r_u\frac{\partial_v}{F} + r_v\frac{\partial_u}{F} = \frac{v}{4M}(\partial_v + r_v\underline{L})$ and $D\omega = \nabla\omega$ for ∇ the induced covariant derivative on Σ_r giving

$$L = (a + b\frac{v_0}{4M}r_v)\underline{L} + \frac{bv_0}{4M}\partial_v - b\nabla\omega.$$

For simplicity we set $A = a + b\frac{v_0}{4M} r_v$ and solve for A, b in $L = A\underline{L} + b(\frac{v_0}{4M}\partial_v) - b\nabla\omega$:

$$2 = \langle L, \underline{L}\rangle = b\langle \frac{v_0}{4M}\partial_v, \underline{L}\rangle = b$$

$$0 = \langle L, L\rangle = 2Ab + b^2|\nabla\omega|^2 = 4(A + |\nabla\!\!\!/\omega|^2)$$

having used $\nabla\omega = \nabla\!\!\!/\omega - |\nabla\!\!\!/\omega|^2\underline{L}$ in the second equality. We conclude that

$$\begin{aligned} L &= \frac{v_0}{2M}\partial_v - |\nabla\!\!\!/\omega|^2\underline{L} - 2(\nabla\!\!\!/\omega - |\nabla\!\!\!/\omega|^2\underline{L}) \\ &= \frac{v_0}{2M}\partial_v + |\nabla\!\!\!/\omega|^2\underline{L} - 2\nabla\!\!\!/\omega. \end{aligned}$$

Lemma 3.2.1. *Given a cross-section $\Sigma := \{r = \omega\}$ of the standard nullcone Ω, given by $\Omega := \{v = v_0\}$ in Kruskal spacetime, we have for the generator $\underline{L}$ satisfying $\underline{L}(r) = 1$ that*

$$\begin{aligned} \langle \tilde{V}, \tilde{W}\rangle &= \omega^2(\tilde{V}, \tilde{W}) \\ \underline{\chi}(\tilde{V}, \tilde{W}) &= \frac{1}{\omega}\langle \tilde{V}, \tilde{W}\rangle \\ \operatorname{tr}\underline{\chi} &= \frac{2}{\omega} \\ \chi(\tilde{V}, \tilde{W}) &= \frac{1}{\omega}(1 - \frac{2M}{\omega} + |\nabla\!\!\!/\omega|^2)\langle \tilde{V}, \tilde{W}\rangle - 2H^\omega(\tilde{V}, \tilde{W}) \\ \operatorname{tr}\chi &= \frac{2}{\omega}\Big(1 - \frac{2M}{\omega} - \omega^2\Delta\!\!\!/\log\omega\Big) \\ \zeta(\tilde{V}) &= -\tilde{V}\log\omega \\ \rho &= \frac{2M}{\omega^3} \end{aligned}$$

where $\tilde{V}, \tilde{W} \in \Gamma(T\Sigma)$ and $(\cdot,\cdot)$ the round metric on $\mathbb{S}^2$.

Proof. The first identity follows trivially from the metric g_K upon restriction to Σ. From the Koszul formula and the fact that $\underline{L}$ is geodesic it is a straight for exercise to show that $D_{\tilde{V}}\underline{L} = \frac{\underline{L}(r)}{r}V|_\Sigma = \frac{1}{\omega}V$. Denoting the Hessian of ω on Σ by H^ω we therefore have

$$
\begin{aligned}
\underline{\chi}(\tilde{V},\tilde{W}) &= \langle D_{\tilde{V}}\underline{L},\tilde{W}\rangle \\
&= \frac{1}{\omega}\langle V,W\rangle \\
\chi(\tilde{V},\tilde{W}) &= \frac{v_0}{2M}\langle D_{\tilde{V}}\partial_v,\tilde{W}\rangle + |\nabla\!\!\!/\omega|^2\langle D_V\underline{L},W\rangle - 2\langle D_{\tilde{V}}\nabla\!\!\!/\omega,\tilde{W}\rangle \\
&= \frac{v_0}{2M}\Big(\langle D_V\partial_v,W\rangle + V\omega\langle D_{\underline{L}}\partial_v,W\rangle + W\omega\langle D_V\partial_v,\underline{L}\rangle + V\omega W\omega\langle D_{\underline{L}}\partial_v,\underline{L}\rangle\Big) \\
&\quad + |\nabla\!\!\!/\omega|^2\underline{\chi}(V,W) - 2H^{\omega}(\tilde{V},\tilde{W}) \\
&= \frac{v_0 r_v}{2M\omega}\langle V,W\rangle + \frac{1}{\omega}|\nabla\!\!\!/\omega|^2\langle V,W\rangle - 2H^{\omega}(\tilde{V},\tilde{W})
\end{aligned}
$$

where in the forth line we use the Koszul formula to evaluate the first term and metric compatibility to show the last three terms vanish. We have

$$\frac{vr_v}{2Mr} = \frac{v}{2Mr}\frac{u}{g'(r)} = \frac{1}{2Mr}\frac{Fg}{4M} = \frac{1}{r}(1-\frac{2M}{r})$$

so the second and forth identities follow upon restriction to Σ. The third identity is simply a trace over Σ of the second. Similarly the fifth follows our taking a trace of the forth and employing the fact that

$$\Delta\!\!\!/\omega - \frac{1}{\omega}|\nabla\!\!\!/\omega|^2 = \omega\Delta\!\!\!/\log\omega.$$

For ζ:

$$
\begin{aligned}
\zeta(\tilde{V}) &= \frac{1}{2}\langle D_{\tilde{V}}\underline{L},L\rangle \\
&= \frac{1}{2\omega}\langle V,\frac{v_0}{2M}\partial_v + |\nabla\!\!\!/\omega|^2\underline{L} - 2\nabla\!\!\!/\omega\rangle \\
&= -\frac{1}{\omega}\langle V,\nabla\!\!\!/\omega\rangle \\
&= -\frac{1}{\omega}\langle\tilde{V},\nabla\!\!\!/\omega\rangle
\end{aligned}
$$

$$= -\tilde{V} \log \omega.$$

From the first identity we conclude that Σ has Gaussian curvature $\mathcal{K} = \frac{1}{\omega^2} - \not{\Delta} \log \omega$ and therefore

$$\langle \vec{H}, \vec{H} \rangle = \operatorname{tr} \chi \operatorname{tr} \underline{\chi} = 4(\mathcal{K} - \frac{2M}{\omega^3}).$$

Since $\zeta - \not{d} \log \operatorname{tr} \underline{\chi} = -\not{d} \log \omega - (-\not{d} \log \omega) = 0$ on Σ we have

$$\rho = \mathcal{K} - \frac{1}{4} \langle \vec{H}, \vec{H} \rangle = \frac{2M}{\omega^3}.$$

□

It follows, in Schwarzschild spacetime, that all foliations to the past of a section of the event horizon $(r = 2M)$ inside the standard nullcone $(v = v_0)$ are (SP)-foliations since $\Sigma = \{r = \omega\}$ satisfies

$$\frac{1}{4} \langle \vec{H}, \vec{H} \rangle - \frac{1}{3} \not{\Delta} \log \rho = \frac{1}{\omega^2}(1 - \frac{2M}{\omega}) > 0 \iff \omega > 2M.$$

Moreover, equality is reached only at the horizon itself indicating physical significance to our property (P). One of the motivating factors for our choice of mass functional comes from our ability, in this special case, to extract the exact mass content within any $\Sigma \subset \Omega$:

$$m(\Sigma) = \frac{1}{2} \Big(\frac{1}{4\pi} \int_\Sigma (\frac{2M}{\omega^3})^{\frac{2}{3}} dA \Big)^{\frac{3}{2}} = \frac{1}{2} \Big(\frac{1}{4\pi} \int_\Sigma \frac{(2M)^{\frac{2}{3}}}{\omega^2} \omega^2 d\mathbb{S}^2 \Big)^{\frac{3}{2}} = M.$$

Lemma 3.2.2. *Suppose Σ is a compact Riemannian manifold, then for any $f \in \mathcal{F}(\Sigma)$*

$$\Big(\int f^{\frac{2}{3}} dA \Big)^{\frac{3}{2}} = \inf_{\psi > 0} \Big(\sqrt{\int \psi^2 dA} \int \frac{|f|}{\psi} dA \Big)$$

Proof. by choosing $\psi_\epsilon^3 = |f| + \epsilon$ for some $\epsilon > 0$ it's a simple verification that

$$\Big(\int (|f|+\epsilon)^{\frac{2}{3}} dA\Big)^{\frac{3}{2}} \geqslant \sqrt{\int \psi_\epsilon^2 dA} \int \frac{|f|}{\psi_\epsilon} dA \geqslant \inf_{\psi>0} \sqrt{\int \psi^2 dA} \int \frac{|f|}{\psi} dA$$

so by the Dominated Convergence Theorem

$$\Big(\int f^{\frac{2}{3}} dA\Big)^{\frac{3}{2}} = \lim_{\epsilon\to 0} \Big(\int (|f|+\epsilon)^{\frac{2}{3}} dA\Big)^{\frac{3}{2}} \geqslant \inf_{\psi>0} \sqrt{\int \psi^2 dA} \int \frac{|f|}{\psi} dA.$$

We show the inequality holds in the opposite direction from Hölder's inequality

$$\int f^{\frac{2}{3}} dA = \int (\frac{f}{\psi})^{\frac{2}{3}} \psi^{\frac{2}{3}} dA \leqslant \Big(\sqrt{\int \psi^2 dA}\Big)^{\frac{2}{3}} \Big(\int \frac{|f|}{\psi} dA\Big)^{\frac{2}{3}}$$

where the result follows from raising both sides to the $\frac{3}{2}$ power and taking an infimum over all $\psi > 0$. □

So given any 2-sphere with non-negative flux $\rho \geqslant 0$ in an arbitrary spacetime, defining $E_H^\psi(\Sigma) := \frac{1}{8\pi}\sqrt{\frac{\int \psi^2 dA}{4\pi}} \int \frac{\rho}{\psi} dA$, we conclude that

$$m(\Sigma) = \inf_{\psi>0} E_H^\psi(\Sigma) \leqslant E_H(\Sigma)$$

as desired. Recalling our use of Hölder's inequality in the proof of Lemma 3.2.2, we see that $m(\Sigma) = E_H(\Sigma)$ if and only if ρ is constant on Σ. So for $\Sigma := \{r = \omega\} \subset \Omega$, where Ω is the standard nullcone in Schwarzschild spacetime, we see that $m(\Sigma)$ underestimates the Hawking energy $E_H(\Sigma)$ with equality only if ρ hence ω is constant. Namely the round spheres within 'time-symmetric' slices given by $t = const > 0$ or equivalently $\frac{v}{u} = const > 0$ (so that $v = v_0$ implies $r = const$) as expected from Sauters work ([25], Lemma 4.4).

3.3 Variation of E_H

In this section we spend some time studying arbitrary normal variations of E_H on admissible spheres following work of Bray, Jauregui and Mars ([7]). The authors of [7] consider 'uniform area expanding flows' according to the flow vector $\partial_s = \vec{I} + \beta\vec{I}^{\perp}$ so we first spend some time extending their Plane Theorem to incorporate arbitrary normal flows $\partial_s = \alpha\vec{I} + \beta\vec{I}^{\perp}$. Subsequently, we show that an arbitrary null flow is obstructed from monotonicity by a term with direct dependence upon ρ in analogy with the variation found by Christodoulou regarding the mass aspect function μ (see, for example, [25] Theorem 4.1). We hope that this points towards ρ being potentially closer to an optimal choice of flux for the Hawking Energy E_H in capturing the ambient spacetime.

The following proposition is known (see [5], Lemma 4), we provide proof to complement the Plane Theorem of [7] and to establish the result in the notation introduced in Definition 3.1.1.

Proposition 3.3.1 (Plane Derivation)**.** *Suppose $\Omega \cong I \times \mathbb{S}^2$, for some interval $I \subset \mathbb{R}$, is a hypersurface of $\mathcal{M}$ and $\alpha \neq 0$ is a smooth function on Ω. Assuming the existence of a foliation of Ω by admissible spheres $\{\Sigma_s\}$ according to the level set function $s : \Omega \to \mathbb{R}$ whereby $\partial_s|_{\Sigma_s} = \alpha\vec{I} = -\alpha\frac{\vec{H}}{H^2}$ then we have*

$$\frac{1}{\sqrt{\frac{|\Sigma_s|}{(16\pi)^3}}}\frac{dE_H}{ds} = \int_{\Sigma_s}(\bar{\alpha} - \alpha)(2\mathcal{K}_s - \frac{1}{2}H^2 - 2\Delta\!\!\!\!/\, \log H)dA$$

$$+ \int \alpha(2G(e_t, e_t) + |\hat{II}_r|^2 + |\hat{II}_t|^2 + 2|\alpha_H|^2 + 2|\nabla\!\!\!\!/ \log H|^2)dA$$

for $II_{r(t)} = \langle \vec{II}, e_{r(t)}\rangle$ where $e_{r(t)}$ is given in Definition 2.1.5.

Before proving Proposition 3.3.1 we will first need to find the second variation of area:

Lemma 3.3.1.

$$\langle \vec{H}, D_{\partial_s}\vec{H}\rangle = -\alpha|II_r|^2 - \alpha Ric_\Omega(e_r, e_r) - H\not\Delta(\frac{\alpha}{H})$$

Proof. In this lemma we temporarily denote the induced covariant derivative on Ω by D noticing that $\langle \vec{H}, D_{\partial_s}\vec{H}\rangle$ calculates the same quantity as if the ambient connection was used. Taking a local basis $\{X_1, X_2\}$ along the foliation we define $\gamma_{ij} := \langle X_i, X_j\rangle$ giving rise to the inverse metric γ^{ij}. For any $V \in \Gamma(T\Omega)$ parallel to the leaves of the foliation (i.e. $V|_{\Sigma_s} = \Gamma(T\Sigma_s)$ we have $[\partial_s, V]s = \partial_s(V(s)) - V(\partial_s(s)) = 0$ giving $[\partial_s, V]|_{\Sigma_s} \in \Gamma(T\Sigma_s)$. As such

$$\begin{aligned}
\langle \vec{H}, D_{\partial_s}\vec{H}\rangle &= \langle \vec{H}, D_{\partial_s}(\gamma^{ij}\vec{\mathrm{II}}_{ij})\rangle \\
&= -\gamma^{ik}\gamma^{jl}(\langle D_{\partial_s}X_k, X_l\rangle + \langle D_{\partial_s}X_l, X_k\rangle)\langle \vec{\mathrm{II}}_{ij}, \vec{H}\rangle \\
&\quad + \gamma^{ij}\langle D_{\partial_s}(D_{X_i}X_j - \not\nabla_{X_i}X_j), \vec{H}\rangle \\
&= -2\gamma^{ik}\gamma^{jl}(\langle [\partial_s, X_k], X_l\rangle - \langle \vec{\mathrm{II}}_{kl}, \partial_s\rangle)\langle \vec{\mathrm{II}}_{ij}, \vec{H}\rangle \\
&\quad + \gamma^{ij}(\langle -R_{\partial_s X_i}X_j + D_{X_i}D_{\partial_s}X_j + D_{[\partial_s, X_i]}X_j - D_{\not\nabla_{X_i}X_j}\partial_s, \vec{H}\rangle
\end{aligned}$$

where we used the fact that $[\partial_s, \not\nabla_{X_i}X_j]|_{\Sigma_s} \in T\Sigma_s$ to get the last term. From the fact that $\partial_s = \alpha\vec{I}$ it follows that

$$\begin{aligned}
2\gamma^{ik}\gamma^{jl}(\langle [\partial_s, X_k], X_l\rangle - \langle \vec{\mathrm{II}}_{kl}, \partial_s\rangle)\langle \vec{\mathrm{II}}_{ij}, \vec{H}\rangle &= 2\gamma^{ik}\langle D_{[\partial_s, X_k]}X_i, \vec{H}\rangle \\
&\quad - 2(-\frac{\alpha}{H^2})\gamma^{ik}\gamma^{jl}\langle \vec{\mathrm{II}}_{kl}, \vec{H}\rangle\langle \vec{\mathrm{II}}_{ij}, \vec{H}\rangle \\
&= 2\gamma^{ik}\langle D_{[\partial_s, X_k]}X_i, \vec{H}\rangle + 2\alpha|\mathrm{II}_r|^2 \\
\gamma^{ij}\langle D_{X_i}D_{\partial_s}X_j + D_{[\partial_s, X_i]}X_j, \vec{H}\rangle &= \gamma^{ij}\langle D_{X_i}[\partial_s, X_j] + D_{X_i}D_{X_j}\partial_s + D_{[\partial_s, X_i]}X_j, \vec{H}\rangle \\
&= \gamma^{ij}\langle [X_i, [\partial_s, X_j]] + D_{[\partial_s, X_j]}X_i + D_{[\partial_s, X_i]}X_j + D_{X_i}D_{X_j}\partial_s, \vec{H}\rangle \\
&= 2\gamma^{ij}\langle D_{[\partial_s, X_i]}X_j, \vec{H}\rangle + \gamma^{ij}\langle D_{X_i}D_{X_j}\partial_s, \vec{H}\rangle
\end{aligned}$$

having used the fact that $[X_i, [\partial_s, X_j]] \in \Gamma(T\Sigma_s)$ to get the final equality. This allows us to simplify to

$$\langle \vec{H}, D_{\partial_s}\vec{H}\rangle = -2\alpha|\mathrm{II}_r|^2 + Ric_{\Omega}(\partial_s, \vec{H}) + \gamma^{ij}\langle (D_{X_i}D_{X_j}\partial_s - D_{\nabla_{X_i}X_j}\partial_s), \vec{H}\rangle.$$

Given also that $\partial_s = \frac{\alpha}{H}e_r$ we see $\langle \partial_s, D_X e_r\rangle = \frac{\alpha}{2H}X\langle e_r, e_r\rangle = 0$ for any $X \in \Gamma(T\Sigma_s)$ so we simplify the last two terms

$$\begin{aligned}
\langle D_{X_i}D_{X_j}\partial_s, \vec{H}\rangle &= -HX_iX_j\langle \partial_s, e_r\rangle + HX_i\langle \partial_s, D_{X_j}e_r\rangle + H\langle D_{X_j}\partial_s, D_{X_i}e_r\rangle \\
&= -HX_iX_j(\frac{\alpha}{H}) + \alpha\langle D_{X_j}e_r, D_{X_i}e_r\rangle \\
&= -HX_iX_j(\frac{\alpha}{H}) + \alpha\gamma^{kl}\langle D_{X_j}e_r, X_k\rangle\langle D_{X_i}e_r, X_l\rangle \\
&= -HX_iX_j(\frac{\alpha}{H}) + \alpha|\,\mathrm{II}_r\,|^2 \\
\langle D_{\nabla_{X_i}X_j}\partial_s, \vec{H}\rangle &= -H\nabla_{X_i}X_j\langle \partial_s, e_r\rangle + H\langle \partial_s, D_{\nabla_{X_i}X_j}e_r\rangle \\
&= -H\nabla_{X_i}X_j(\frac{\alpha}{H})
\end{aligned}$$

and the result follows after we collect all the terms and take a trace over i, j. □

Proof. (Proposition 3.3.1) The proof follows in parallel to the Plane Theorem of [7] (Theorem 2.1). From the first variation of area formula:

$$\begin{aligned}
d\dot{A}_s &= -\langle \vec{H}, \partial_s\rangle dA_s = \alpha dA_s \\
\implies |\dot{\Sigma_s}| &= |\Sigma_s|\bar{\alpha}(s).
\end{aligned}$$

So variation of the Hawking Energy gives:

$$\begin{aligned}
\frac{dE_H}{ds} &= \frac{d}{ds}\Big(\sqrt{\frac{|\Sigma_s|}{(16\pi)^3}}\Big(16\pi - \int H^2 dA_s\Big)\Big) \\
&= \sqrt{\frac{|\Sigma_s|}{(16\pi)^3}}\Big(\frac{1}{2}\bar{\alpha}\Big(16\pi - \int H^2 dA_s\Big) - 2\int \langle \vec{H}, D_{\partial_s}\vec{H}\rangle dA_s - \int \alpha H^2 dA_s\Big) \\
&= \sqrt{\frac{|\Sigma_s|}{(16\pi)^3}}\Big(\int \bar{\alpha}(2\mathcal{K}_s - \frac{1}{2}H^2)dA_s + \int 2\alpha|\mathrm{II}_r|^2 + 2\alpha Ric_{\Omega}(e_r, e_r) + 2H\Delta(\frac{\alpha}{H})dA
\end{aligned}$$

$$- \int \alpha H^2 dA \Big)$$

where we used the Gauss-Bonnet Theorem and Proposition 3.3.1 respectively to get the first and second integrands of the last line. As in [7] we now trace the Gauss equation for Σ_s in Ω twice over Σ_s to get

$$2Ric_\Omega(e_r, e_r) = S - 2\mathcal{K}_s + H^2 - |\mathrm{II}_r|^2$$

for S the scalar curvature of Ω. We then trace the Gauss equation for Ω in $\mathcal{M}$ twice over Ω to conclude

$$S = 2G(e_t, e_t) + 2|\alpha_H|^2 + |\mathrm{II}_t|^2$$

since $e_t \in \Gamma(T^\perp\Omega)$. Substitution into our variation of E_H therefore gives us after some algebraic manipulation that

$$\frac{dE_H}{ds} = \sqrt{\frac{|\Sigma_s|}{(16\pi)^3}} \Big(\int (\bar{\alpha} - \alpha)(2\mathcal{K}_s - \frac{1}{2}H^2) dA_s$$

$$+ \int 2G(e_t, e_t) + (|\mathrm{II}_r|^2 - \frac{1}{2}H^2) + |\mathrm{II}_t|^2 + 2|\alpha_H|^2 + 2H \not{\Delta}(\frac{\alpha}{H}) dA_s \Big).$$

First performing an integration by parts on the last term

$$\int H \not{\Delta}(\frac{\alpha}{H}) dA = \int (\not{\Delta} H) \frac{\alpha}{H} dA$$

followed by the identity $\frac{\not{\Delta} H}{H} = \not{\Delta} \log H + |\not{\nabla} \log H|^2$ we obtain the first line of the variation in Proposition 3.3.1. The second follows from the fact that

$$|\mathrm{II}_r|^2 = |\hat{\mathrm{II}}_r|^2 + \frac{1}{2}H^2$$

$$\mathrm{II}_t = \hat{\mathrm{II}}_t$$

□

We refer the reader to [7] (Theorem 2.2) for proof of the Cylinder Theorem:

Proposition 3.3.2. *Under the same hypotheses as in Proposition 3.3.1 with* $\partial_s = \beta \vec{I}^{\perp}$ *for some smooth function* $\beta \neq 0$ *on* Ω *and* $\vec{I}^{\perp} = \frac{e_t}{H}$ *we have*

$$\frac{1}{\sqrt{\frac{|\Sigma_s|}{(16\pi)^3}}} \frac{dE_H}{ds} = \int_{\Sigma_s} \beta(2G(e_t, e_r) + 2\langle \hat{I\!I}_r, \hat{I\!I}_t \rangle + 4\alpha_H(\nabla\!\!\!/ \log H) + 2\nabla\!\!\!/ \cdot \alpha_H) dA_s.$$

The full variation of E_H is known from ([5], Lemma 3), we are now in a position to show it within our context:

Corollary 3.3.2.1. *Under the same hypotheses as Proposition 3.3.1 and 3.3.2 with* $\partial_s = \alpha \vec{I} + \beta \vec{I}^{\perp}$

$$\frac{1}{\sqrt{\frac{|\Sigma_s|}{(16\pi)^3}}} \frac{dE_H}{ds} = \int_{\Sigma_s} (\bar{\alpha} - \alpha)(2\mathcal{K}_s - \frac{1}{2}H^2 - 2\Delta\!\!\!/ \log H) dA_s$$

$$+ \int_{\Sigma_s} \alpha(2G(e_t, e_t) + |\hat{I\!I}_r|^2 + |\hat{I\!I}_t|^2 + 2|\alpha_H|^2 + 2|\nabla\!\!\!/ \log H|^2) dA_s$$

$$+ \int_{\Sigma_s} \beta(2G(e_t, e_r) + 2\langle \hat{I\!I}_r, \hat{I\!I}_t \rangle + 4\alpha_H(\nabla\!\!\!/ \log H) + 2\nabla\!\!\!/ \cdot \alpha_H) dA_s$$

Proof. As in [7] (Theorem 1.13) variation of E_H is achieved by summing the contributions from Propositions 3.3.1 and 3.3.2 since the variation of the area form and the mean curvature vector are known to be $\mathbb{R}$-linear over the flow vector decomposition. □

Subsequently, we achieve an arbitrary past(future) directed null flow by setting $\alpha = \mp\beta > 0$ in Corollary 3.3.2.1 giving $\partial_s = \alpha(\vec{I} \mp \vec{I}^{\perp})$ and

$$\frac{1}{\sqrt{\frac{|\Sigma_s|}{(16\pi)^3}}} \frac{dE_H}{ds} = \int_{\Sigma_s} (\bar{\alpha} - \alpha)(2\mathcal{K}_s - \frac{1}{2}H^2 \pm 2\nabla\!\!\!/ \cdot \alpha_H - 2\Delta\!\!\!/ \log H) dA_s$$

$$+ \int_{\Sigma_s} \alpha(2G(e_t, e_t \mp e_r) + |\hat{\mathrm{I\!I}}_r \mp \hat{\mathrm{I\!I}}_t|^2 + 2|\alpha_H \mp \nabla\!\!\!/ \log H|^2) dA_s.$$

It follows in an energy dominated spacetime that the only obstruction to a non-decreasing Hawking energy is the integrand

$$(\bar{\alpha} - \alpha)(2\mathcal{K}_s - \frac{1}{2}H^2 \pm 2\nabla\!\!\!/ \cdot \alpha_H - 2\Delta\!\!\!/ \log H) = 2(\bar{\alpha} - \alpha)\rho_{\mp}.$$

In particular $\frac{dE_H}{ds} \geqslant 0$ for any foliation where ρ is constant on each Σ_s, moreover, since $m_{\mp}(\Sigma_s) = E_H(\Sigma_s)$ in this case (provided also $\rho_{\mp} \geqslant 0$) we have monotonicity of our quasi local mass as well. We extend beyond this case in the next section with the proof of Theorem 2.1.1.

4

Propagation of ρ

In this chapter we work towards proving Theorem 2.1.1 by finding the propagation of our flux function ρ along an arbitrary null flow.

4.1 Setup

We adopt the same setup as in [18] which we summarize here in order to introduce our notation:

Suppose Ω is a smooth connected, null hypersurface embedded in $(\mathcal{M}, \langle \cdot, \cdot \rangle)$. Here we let $\underline{L}$ be a smooth, non-vanishing, null vector field of Ω, $\underline{L} \in \Gamma(T\Omega)$. It's a well known fact (see, for example, [11]) that the integral curves of $\underline{L}$ are pre-geodesic so we're able to find $\kappa \in \mathcal{F}(\Omega)$ such that $D_{\underline{L}}\underline{L} = \kappa \underline{L}$.

We assume the existence of an embedded sphere Σ in Ω such that any integral curve of $\underline{L}$ intersects Σ precisely once. As previously used, we will refer to such Σ as *cross-sections* of Ω. This gives rise to a natural submersion $\pi : \Omega \to \Sigma$ sending $p \in \Omega$ to the intersection with Σ of the integral curve $\gamma_p^{\underline{L}}$ of $\underline{L}$ for which $\gamma_p^{\underline{L}}(0) = p$. Given $\underline{L}$ and a constant s_0 we may construct a function $s \in \mathcal{F}(\Omega)$ from $\underline{L}(s) = 1$ and $s|_\Sigma = s_0$. For $q \in \Sigma$, if $(s_-(q), s_+(q))$ represents the range of s along $\gamma_q^{\underline{L}}$ then

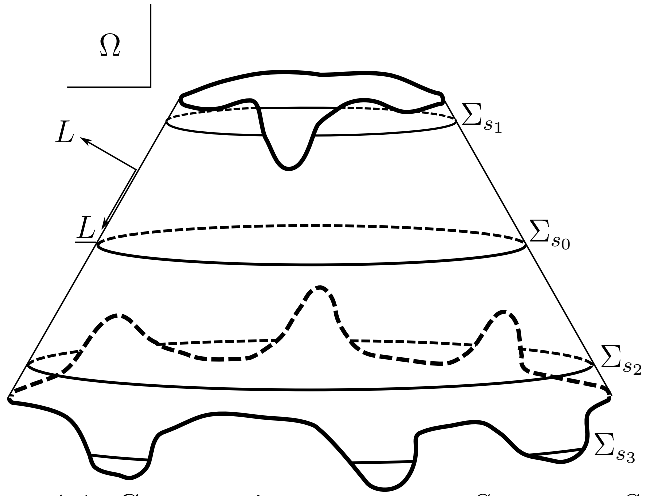

FIGURE 4.1: Cross-sections: $s_3 < s_2 < S_- < s_0 < S_+ < s_1$

letting $S_- = \sup_\Sigma s_-$ and $S_+ = \inf_\Sigma s_+$ we notice that the interval (S_-, S_+) is non-empty. Given that $\underline{L}(s) = 1$ the Implicit Function Theorem gives for $t \in (S_-, S_+)$ that $\Sigma_t := \{p \in \Omega | s(p) = t\}$ is diffeomorphic to $\mathbb{S}^2$ through Σ. For $s < S_-$ or $s > S_+$, in the case that Σ_s is non-empty, although smooth it may no longer be connected. We have that the collection $\{\Sigma_s\}$ gives a foliation of Ω.

We construct another null vector field L by assigning at every $p \in \Omega$ $L|p \in T_p\mathcal{M}$ be the unique null vector satisfying $\langle \underline{L}, L\rangle = 2$ and $\langle L, v\rangle = 0$ for any $v \in T_p\Sigma_{s(p)}$. As before each Σ_s is endowed with an induced metric γ_s, two null second fundamental forms $\underline{\chi} = -\langle \vec{\mathrm{II}}, \underline{L}\rangle$ and $\chi = -\langle \vec{\mathrm{II}}, L\rangle$ as well as the connection 1-form (or torsion) $\zeta(V) = \frac{1}{2}\langle D_V \underline{L}, L\rangle$. We will need the following known result ([25]):

Lemma 4.1.1. *Given $V \in \Gamma(T\Sigma_s)$,*

- $D_V \underline{L} = \vec{\underline{\chi}}(V) + \zeta(V)\underline{L}$
- $D_V L = \vec{\chi}(V) - \zeta(V)L$
- $D_{\underline{L}} L = -2\vec{\zeta} - \kappa L$

where, given $V, W \in \Gamma(T\Sigma)$, the vector fields $\vec{\zeta}$, $\vec{\underline{\chi}}(V)$ are uniquely determined by $\langle \vec{\zeta}, V\rangle = \zeta(V)$ and $\langle \vec{\underline{\chi}}(V), W\rangle = \underline{\chi}(V, W)$.

Proof. It suffices to check all identities agree by taking the metric inner product with vectors $\underline{L}, L$ and an extension W satisfying $W|_{\Sigma_s} \in \Gamma(T\Sigma_s)$ keeping in mind that $[\underline{L}, W]|_{\Sigma_s} \in \Gamma(T\Sigma_s)$. We leave this verification to the reader. $\square$

For any cross-section Σ of Ω and $v \in T_q(\Sigma)$ we may extend v along the generator $\gamma_q^{\underline{L}}$ according to

$$\dot{V}(s) = D_{V(s)}\underline{L}$$
$$V(0) = v.$$

Since $x \in T_p\Omega \iff \langle \underline{L}|_p, x\rangle = 0$ we see from the fact that

$$(\langle V(\dot{s}), \underline{L}\rangle) = \langle D_{V(s)}\underline{L}, \underline{L}\rangle + \kappa\langle V(s), \underline{L}\rangle = \frac{1}{2}V(s)\langle \underline{L}, \underline{L}\rangle + \kappa\langle V(s), \underline{L}\rangle = \kappa\langle V(s), \underline{L}\rangle,$$

and $\langle V(0), \underline{L}\rangle = 0$, that infact $\langle V(s), \underline{L}\rangle = 0$ for all s. As a result, any section $W \in \Gamma(T\Sigma)$ is extended to all of Ω satisfying $[\underline{L}, W] = 0$. We also notice along each generator $0 = [\underline{L}, W]s = \underline{L}(Ws) = \dot{W}s$ such that $Ws|_\Sigma = 0$ forces $Ws = 0$ on all of Ω. We conclude that $W|_{\Sigma_s} \in \Gamma(T\Sigma_s)$ and denote by $E(\Sigma) \subset \Gamma(T\Omega)$ the set of such extensions off of Σ along $\underline{L}$. We also note that linear independence is preserved along generators by standard uniqueness theorems allowing us to extend basis fields $\{X_1, X_2\} \subset \Gamma(T\Sigma)$ off of Σ as well.

4.2 The Structure Equations

We will need to propagate the Christoffel symbols with the known result ([25]):

Lemma 4.2.1. *Given $U, V, W \in E(\Sigma)$,*

$$\langle [\underline{L}, \nabla\!\!\!/_V W], U\rangle = (\nabla\!\!\!/_V \underline{\chi})(W, U) + (\nabla\!\!\!/_W \underline{\chi})(V, U) - (\nabla\!\!\!/_U \underline{\chi})(V, W)$$

where $\nabla\!\!\!/$ the induced covariant derivative on each Σ_s.

Proof. Starting from the Koszul formula

$$2\langle \not\nabla_V W, U\rangle = V\langle W, U\rangle + W\langle V, U\rangle - U\langle V, W\rangle - \langle V, [W, U]\rangle + \langle W, [U, V]\rangle + \langle U, [V, W]\rangle$$

we apply $\underline{L}$ to the left hand term to get

$$\underline{L}\langle \not\nabla_V W, U\rangle = \langle D_{\underline{L}} \not\nabla_V W, U\rangle + \langle \not\nabla_V W, D_{\underline{L}} U\rangle = \langle [\underline{L}, \not\nabla_V W], U\rangle + 2\underline{\chi}(\not\nabla_V W, U)$$

and to the right keeping in mind that $[V, W] \in E(\Sigma)$

$$\begin{aligned}
&\underline{L}\Big(V\langle W, U\rangle + W\langle V, U\rangle - U\langle V, W\rangle - \langle V, [W, U]\rangle + \langle W, [U, V]\rangle + \langle U, [V, W]\rangle\Big) \\
&\qquad = V\underline{L}\langle W, U\rangle + W\underline{L}\langle V, U\rangle - U\underline{L}\langle V, W\rangle - 2\underline{\chi}(V, [W, U]) \\
&\qquad\quad + 2\underline{\chi}(W, [U, V]) + 2\underline{\chi}(U, [V, W]) \\
&\qquad = 2\Big(V\underline{\chi}(W, U) + W\underline{\chi}(V, U) - U\underline{\chi}(V, W) - \underline{\chi}(V, [W, U]) \\
&\qquad\quad + \underline{\chi}(W, [U, V]) + \underline{\chi}(U, [V, W])\Big) \\
&\qquad = 2\Big((\not\nabla_V \underline{\chi})(W, U) + (\not\nabla_W \underline{\chi})(V, U) - (\not\nabla_U \underline{\chi})(V, W) + 2\underline{\chi}(\not\nabla_V W, U)\Big).
\end{aligned}$$

Equating terms according to the Koszul formula the result follows upon cancellation of the term $\underline{\chi}(\not\nabla_V W, U)$. □

Lemma 4.2.2. *For $S, U, V, W \in E(\Sigma)$,*

$$\begin{aligned}
\langle [\underline{L}, \not R_{VW} U], S\rangle &= (\not\nabla_W \not\nabla_V \underline{\chi})(U, S) - (\not\nabla_V \not\nabla_W \underline{\chi})(U, S) + (\not\nabla_W \not\nabla_U \underline{\chi})(V, S) \\
&\quad - (\not\nabla_V \not\nabla_U \underline{\chi})(W, S) + (\not\nabla_V \not\nabla_S \underline{\chi})(W, U) - (\not\nabla_W \not\nabla_S \underline{\chi})(V, U)
\end{aligned}$$

where $\not R$ the induced Riemann curvature tensor on Σ_s.

Proof. We notice any $f \in \mathcal{F}(\Sigma)$ can be extended to all of Ω by imposing $\underline{L}(f) = 0$ along generators. As such $fV \in E(\Sigma)$ and

$$[\underline{L}, \not R_{fVW} U] = [\underline{L}, \not R_{VfW} U] = [\underline{L}, \not R_{VW} fU] = [\underline{L}, f\not R_{VW} U] = f[\underline{L}, \not R_{VW} U].$$

Within $E(\Sigma)$ we conclude that both $\langle[\underline{L}, \not{R}_{VW}U], S\rangle$ and the right hand side of the identity restricts to 4-tensors pointwise on each Σ_s. It therefore suffices to prove the identity pointwise. In particular, for any $v, w \in T_q\Sigma_s$ we extend to vector fields $V, W \in E(\Sigma)$ such that $\not\nabla_V W|_q = 0$. The Riemann tensor on Σ_s reads

$$
\begin{aligned}
\langle \not{R}_{VW}U, S\rangle &= \langle \not\nabla_{[V,W]}U, S\rangle - \langle \not\nabla_V \not\nabla_W U, S\rangle + \langle \not\nabla_W \not\nabla_V U, S\rangle \\
&= \langle \not\nabla_{[V,W]}U, S\rangle - V\langle \not\nabla_W U, S\rangle + \langle \not\nabla_W U, \not\nabla_V S\rangle + W\langle \not\nabla_V U, S\rangle \\
&\quad - \langle \not\nabla_V U, \not\nabla_W S\rangle
\end{aligned}
$$

so applying $\underline{L}$ to the terms on the right assuming restriction to $q \in \Sigma_s$ we have

$$
\begin{aligned}
&\underline{L}\Big(\langle \not\nabla_{[V,W]}U, S\rangle - V\langle \not\nabla_W U, S\rangle + \langle \not\nabla_W U, \not\nabla_V S\rangle + W\langle \not\nabla_V U, S\rangle - \langle \not\nabla_V U, \not\nabla_W S\rangle\Big) \\
&= \langle[\underline{L}, \not\nabla_{[V,W]}U], S\rangle - V\underline{L}\langle \not\nabla_W U, S\rangle + W\underline{L}\langle \not\nabla_V U, S\rangle \\
&= -V\langle[\underline{L}, \not\nabla_W U], S\rangle - 2V\underline{\chi}(\not\nabla_W U, S) + W\langle[\underline{L}, \not\nabla_V U], S\rangle + 2W\underline{\chi}(\not\nabla_V U, S)
\end{aligned}
$$

where the first term in the second line vanishes as a result of Lemma 4.2.1 since $[V, W] \in E(\Sigma)$ and $[V, W]|_q = 0$. Using Lemma 4.2.1 on the first and third terms of the third line we get

$$
\begin{aligned}
&= -V\Big((\not\nabla_W\underline{\chi})(U, S) + (\not\nabla_U\underline{\chi})(W, S) - (\not\nabla_S\underline{\chi})(W, U)\Big) - 2V\underline{\chi}(\not\nabla_W U, S) \\
&\qquad + W\Big((\not\nabla_V\underline{\chi})(U, S) + (\not\nabla_U\underline{\chi})(V, S) - (\not\nabla_S\underline{\chi}(V, U)\Big) \\
&\qquad + 2W\underline{\chi}(\not\nabla_V U, S) \\
&= -(\not\nabla_V\not\nabla_W\underline{\chi})(U, S) - (\not\nabla_V\not\nabla_U\underline{\chi})(W, S) + (\not\nabla_V\not\nabla_S\underline{\chi})(W, U) - 2V\underline{\chi}(\not\nabla_W U, S) \\
&\qquad + (\not\nabla_W\not\nabla_V\underline{\chi})(U, S) + (\not\nabla_W\not\nabla_U\underline{\chi})(V, S) - (\not\nabla_W\not\nabla_S\underline{\chi})(V, U) \\
&\qquad + 2W\underline{\chi}(\not\nabla_V U, S).
\end{aligned}
$$

We also note that restriction to $q \in \Sigma_s$ gives

$$
0 = (\not\nabla_V\underline{\chi})(\not\nabla_W U, S) = V\underline{\chi}(\not\nabla_W U, S) - \underline{\chi}(\not\nabla_V\not\nabla_W U, S)
$$

allowing us to simplify the remaining terms above

$$-2V\underline{\chi}(\nabla\!\!\!/_W U, S) + 2W\underline{\chi}(\nabla\!\!\!/_V U, S) = 2\Big(-\underline{\chi}(\nabla\!\!\!/_V \nabla\!\!\!/_W U, S) + \underline{\chi}(\nabla\!\!\!/_W \nabla\!\!\!/_V U, S)\Big)$$

$$= 2\underline{\chi}(R\!\!\!/_{VW} U, S).$$

Since

$$\underline{L}\langle R\!\!\!/_{VW} U, S\rangle = \langle [\underline{L}, R\!\!\!/_{VW} U], S\rangle + 2\underline{\chi}(R\!\!\!/_{VW} U, S)$$

the result follows upon cancellation of $2\underline{\chi}(R\!\!\!/_{VW} U, S)$ given that q was arbitrarily chosen. □

Now we're in a position to find the structure equations that we'll need to propagate ρ. Recalling that the tensors γ_s, $\underline{\chi}$, χ and ζ are restrictions of associated tensors on Ω we measure their propagation with the Lie derivative along $\underline{L}$. The following proposition is known ([25],[11]), we provide proof for completeness:

Proposition 4.2.1 (Structure Equations)**.**

$$\underline{L}\mathcal{K} = -\operatorname{tr}\underline{\chi}\mathcal{K} - \frac{1}{2}\Delta\!\!\!/ \operatorname{tr}\underline{\chi} + \nabla\!\!\!/ \cdot (\nabla\!\!\!/ \cdot \hat{\underline{\chi}}) \tag{4.1}$$

$$\mathcal{L}_{\underline{L}}\gamma = 2\underline{\chi} \tag{4.2}$$

$$\mathcal{L}_{\underline{L}}\underline{\chi} = -\underline{\alpha} + \frac{1}{2}|\hat{\underline{\chi}}|^2\gamma + \operatorname{tr}\underline{\chi}\hat{\underline{\chi}} + \frac{1}{4}(\operatorname{tr}\underline{\chi})^2\gamma + \kappa\underline{\chi} \tag{4.3}$$

$$\underline{L}\operatorname{tr}\underline{\chi} = -\frac{1}{2}(\operatorname{tr}\underline{\chi})^2 - |\hat{\underline{\chi}}|^2 - G(\underline{L}, \underline{L}) + \kappa\operatorname{tr}\underline{\chi} \tag{4.4}$$

$$\begin{aligned}\mathcal{L}_{\underline{L}}\chi = {} & \Big(\mathcal{K} + \hat{\underline{\chi}}\cdot\hat{\chi} + \frac{1}{2}G(\underline{L}, L)\Big)\gamma + \frac{1}{2}\operatorname{tr}\underline{\chi}\hat{\chi} + \frac{1}{2}\operatorname{tr}\chi\hat{\underline{\chi}} \\ & - \hat{G} - 2S(\nabla\!\!\!/\zeta) - 2\zeta\otimes\zeta - \kappa\chi\end{aligned} \tag{4.5}$$

$$\underline{L}\operatorname{tr}\chi = G(\underline{L}, L) + 2\mathcal{K} - 2\nabla\!\!\!/\cdot\zeta - 2|\zeta|^2 - \langle\vec{H}, \vec{H}\rangle - \kappa\operatorname{tr}\chi \tag{4.6}$$

$$\mathcal{L}_{\underline{L}}\zeta = G_{\underline{L}} - \nabla\!\!\!/\cdot\hat{\underline{\chi}} - \operatorname{tr}\underline{\chi}\zeta + \frac{1}{2}d\!\!\!/\operatorname{tr}\underline{\chi} + d\!\!\!/\kappa \tag{4.7}$$

where $\underline{\alpha}$ is the symmetric 2-tensor given by $\underline{\alpha}(V, W) = \langle R_{\underline{L}V}\underline{L}, W\rangle$, $S(T)$ represents the symmetric part of a 2-tensor T, $G_{\underline{L}} = G(\underline{L}, \cdot)|_{\Sigma_s}$ and $\hat{G} = G|_{\Sigma_s} - \frac{1}{2}(\operatorname{tr}_\gamma G)\gamma$.

Proof. We prove each equation in turn, when used we will assume $S, U, V, W \in E(\Sigma)$:

1. Since Σ is of dimension two we have

$$\mathcal{K}\{\langle V, U\rangle\langle W, S\rangle - \langle V, S\rangle\langle W, U\rangle\} = \langle \not{R}_{VW}U, S\rangle$$

Applying $\underline{L}$ to the left hand side of the equality we get

$$(\underline{L}\mathcal{K})\{\langle V, U\rangle\langle W, S\rangle - \langle V, S\rangle\langle W, U\rangle\} + 2\mathcal{K}\{\langle W, S\rangle\underline{\chi}(V, U) + \langle V, U\rangle\underline{\chi}(W, S) - \langle W, U\rangle\underline{\chi}(V, S) - \langle V, S\rangle\underline{\chi}(W, U)\}$$

so that a trace over V, U and then W, S gives

$$2\underline{L}\mathcal{K} + 4\operatorname{tr}\underline{\chi}\mathcal{K}.$$

Applying $\underline{L}$ to the right hand side we have

$$\underline{L}\langle \not{R}_{VW}U, S\rangle = \langle [\underline{L}, \not{R}_{VW}U], S\rangle + 2\underline{\chi}(\not{R}_{VW}U, S)$$

allowing us to use Lemma 4.2.2. Taking a trace over V, U and W, S we get

$$2\not{\nabla}\cdot\not{\nabla}\cdot\underline{\chi} - 2\not{\Delta}\operatorname{tr}\underline{\chi} + 2\operatorname{tr}\underline{\chi}K$$

having used the fact that $\not{R}ic = K\gamma$ in obtaining the last term. Equating terms we conclude that

$$\underline{L}K = \not{\nabla}\cdot\not{\nabla}\cdot\underline{\chi} - \not{\Delta}\operatorname{tr}\underline{\chi} - \operatorname{tr}\underline{\chi}K = \not{\nabla}\cdot\not{\nabla}\cdot\hat{\underline{\chi}} - \frac{1}{2}\not{\Delta}\operatorname{tr}\underline{\chi} - \operatorname{tr}\underline{\chi}K$$

2. Coming from Lemma 4.1.1 we have already made extensive use of this identity:

$$\begin{aligned}(\mathcal{L}_{\underline{L}}\gamma)(V, W) &= \underline{L}\langle V, W\rangle = \langle D_{\underline{L}}V, W\rangle + \langle V, D_{\underline{L}}W\rangle \\ &= \langle D_V\underline{L}, W\rangle + \langle V, D_W\underline{L}\rangle \\ &= 2\underline{\chi}(V, W)\end{aligned}$$

3. $(\mathcal{L}_{\underline{L}}\underline{\chi})(V,W) = \underline{L}\underline{\chi}(V,W) = \underline{L}\langle D_V\underline{L}, W\rangle$

$$= \langle D_{\underline{L}}D_V\underline{L}, W\rangle + \langle D_V\underline{L}, D_{\underline{L}}W\rangle$$

$$= \langle R_{V\underline{L}}\underline{L} + D_VD_{\underline{L}}\underline{L}, W\rangle + \langle \vec{\underline{\chi}}(V) + \zeta(V)\underline{L}, \vec{\underline{\chi}}(W) + \zeta(W)\underline{L}\rangle$$

$$= -\langle R_{\underline{L}V}\underline{L}, W\rangle + \kappa\underline{\chi}(V,W) + \langle \vec{\underline{\chi}}(V), \vec{\underline{\chi}}(W)\rangle$$

having used Lemma 4.1.1 to get the third line. Since $\vec{\underline{\chi}}(V) = \vec{\hat{\underline{\chi}}}(V) + \frac{1}{2}\operatorname{tr}\underline{\chi}V$ we see that

$$\langle \vec{\underline{\chi}}(V), \vec{\underline{\chi}}(W)\rangle = \langle \vec{\hat{\underline{\chi}}}(V), \vec{\hat{\underline{\chi}}}(W)\rangle + \operatorname{tr}\underline{\chi}\hat{\underline{\chi}}(V,W) + \frac{1}{4}(\operatorname{tr}\underline{\chi})^2\langle V,W\rangle$$

$$= \frac{1}{2}|\hat{\underline{\chi}}|^2\langle V,W\rangle + \operatorname{tr}\underline{\chi}\hat{\underline{\chi}}(V,W) + \frac{1}{4}(\operatorname{tr}\underline{\chi})^2\langle V,W\rangle$$

using the fact that $AB + BA = tr(AB)\mathbb{I}$ for traceless symmetric 2×2 matrices to get the second line. The result follows.

4. We will denote tensor contraction between the contravariant a-th and covariant b-th slots by C^a_b. Extending a local basis off of Σ and applying Gram-Schmidt we get an orthonormal frame field $\{E_1, E_2\}$ allowing us to write

$$g^{-1}|_\Omega = \gamma^{-1} = E_1\otimes E_1 + E_2\otimes E_2$$

and $\gamma = E_1^\flat\otimes E_1^\flat + E_2^\flat\otimes E_2^\flat$. It's an easy exercise to show $C^2_1\gamma^{-1}\otimes\gamma = \delta - \underline{L}\otimes ds$ whereby $\delta(\eta, X) = \eta(X)$ for any 1-form η and vector field X. Since δ and $\underline{L}\otimes ds$ are Lie constant along $\underline{L}$

$$0 = \mathcal{L}_{\underline{L}}C^2_1\gamma^{-1}\otimes\gamma = C^2_1(\mathcal{L}_{\underline{L}}\gamma^{-1}\otimes\gamma + \gamma^{-1}\otimes 2\underline{\chi})$$

giving

$$-2C^2_1C^2_1\gamma^{-1}\otimes\underline{\chi}\otimes\gamma^{-1} = -C^2_1(C^2_1\gamma^{-1}\otimes 2\underline{\chi})\otimes\gamma^{-1}$$

$$= C^2_1(C^2_1\mathcal{L}_{\underline{L}}\gamma^{-1}\otimes\gamma)\otimes\gamma^{-1}$$

$$= C^2_1\mathcal{L}_{\underline{L}}\gamma^{-1}\otimes(C^1_2\gamma\otimes\gamma^{-1})$$

$$= C_1^2 \mathcal{L}_{\underline{L}} \gamma^{-1} \otimes (E_1^\flat \otimes E_1 + E_2^\flat \otimes E_2)$$

$$= \mathcal{L}_{\underline{L}} \gamma^{-1}.$$

As a result

$$\begin{aligned}
\underline{L} \operatorname{tr} \underline{\chi} &= \mathcal{L}_{\underline{L}} C_1^1 C_2^2 \gamma^{-1} \otimes \underline{\chi} \\
&= C_1^1 C_2^2 (\mathcal{L}_{\underline{L}} \gamma^{-1} \otimes \underline{\chi} + \gamma^{-1} \otimes \mathcal{L}_{\underline{L}} \underline{\chi}) \\
&= -2|\underline{\chi}|^2 + \operatorname{tr} \mathcal{L}_{\underline{L}} \underline{\chi} \\
&= -2|\underline{\chi}|^2 - Ric(\underline{L}, \underline{L}) + \kappa \operatorname{tr} \underline{\chi} + |\underline{\chi}|^2 \\
&= -(\hat{\underline{\chi}} + \frac{1}{2} \operatorname{tr} \underline{\chi} \gamma) \cdot (\hat{\underline{\chi}} + \frac{1}{2} \operatorname{tr} \underline{\chi} \gamma) - G(\underline{L}, \underline{L}) + \kappa \operatorname{tr} \underline{\chi} \\
&= -\frac{1}{2} (\operatorname{tr} \underline{\chi})^2 - |\hat{\underline{\chi}}|^2 - G(\underline{L}, \underline{L}) + \kappa \operatorname{tr} \underline{\chi}
\end{aligned}$$

5. $$\begin{aligned}
(\mathcal{L}_{\underline{L}} \chi)(V, W) &= \underline{L} \langle D_V L, W \rangle = \langle D_{\underline{L}} D_V L, W \rangle + \langle D_V L, D_{\underline{L}} W \rangle \\
&= \langle R_{V \underline{L}} L + D_V D_{\underline{L}} L, W \rangle + \langle \vec{\chi}(V) - \zeta(V) L, \vec{\underline{\chi}}(W) + \zeta(W) \underline{L} \rangle \\
&= \langle R_{V \underline{L}} L, W \rangle + V \langle -2\vec{\zeta} - \kappa L, W \rangle + \langle 2\vec{\zeta} + \kappa L, D_V W \rangle \\
&\quad + \langle \vec{\chi}(V), \vec{\underline{\chi}}(W) \rangle - 2\zeta(V)\zeta(W) \\
&= \langle R_{V \underline{L}} L, W \rangle - 2(\nabla\!\!\!/_V \zeta)(W) - \kappa \chi(V, W) + \langle \vec{\chi}(V), \vec{\underline{\chi}}(W) \rangle \\
&\quad - 2\zeta(V)\zeta(W).
\end{aligned}$$

Having used Lemma 4.1.1 to get the second and third lines.

Lemma 4.2.3. *The first term satisfies the identity*

$$\begin{aligned}
\langle R_{V \underline{L}} W, L \rangle = &-\Big(\mathcal{K} - \frac{1}{4} \langle \vec{H}, \vec{H} \rangle + \frac{1}{2} \hat{\underline{\chi}} \cdot \hat{\chi} + \frac{1}{2} G(\underline{L}, L)\Big) \langle V, W \rangle + \hat{G}(V, W) \\
&- (curl \zeta)(V, W) + \frac{1}{2} \Big(\langle \vec{\chi}(V), \vec{\underline{\chi}}(W) \rangle - \langle \vec{\underline{\chi}}(V), \vec{\chi}(W) \rangle \Big)
\end{aligned}$$

Proof. From the first Bianchi identity followed by the Ricci equation ([21], pg.125)

$$\langle R_{V\underline{L}}W, L\rangle + \langle R_{\underline{L}W}V, L\rangle = \langle R_{VW}\underline{L}, L\rangle$$

$$= \langle \not{R}^{\perp}_{VW}\underline{L}, L\rangle + \langle \tilde{\mathrm{II}}(V, L), \tilde{\mathrm{II}}(W, \underline{L})\rangle - \langle \tilde{\mathrm{II}}(V, \underline{L}), \tilde{\mathrm{II}}(W, L)\rangle$$

where using Lemma 4.1.1

$$\langle \not{R}^{\perp}_{VW}\underline{L}, L\rangle := \langle D^{\perp}_{[V,W]}\underline{L} - [D^{\perp}_{V}, D^{\perp}_{W}]\underline{L}, L\rangle = -2(\not{\nabla}_V\zeta)(W) + 2(\not{\nabla}_W\zeta)(V)$$

$$= -2(\mathrm{curl}\zeta)(V, W)$$

$$\tilde{\mathrm{II}}(V, \underline{L}) := D^{||}_{V}\underline{L} = \underline{\vec{\chi}}(V)$$

$$\tilde{\mathrm{II}}(V, L) := D^{||}_{V}L = \vec{\chi}(V).$$

We conclude that the antisymmetric part satisfies

$$\frac{1}{2}\Big(\langle R_{V\underline{L}}W, L\rangle - \langle R_{W\underline{L}}V, L\rangle\Big) = -(\mathrm{curl}\zeta)(V, W) + \frac{1}{2}\Big(\langle \vec{\chi}(V), \underline{\vec{\chi}}(W)\rangle - \langle \underline{\vec{\chi}}(V), \vec{\chi}(W)\rangle\Big).$$

Next we find that

$$G(V, W) = \mathrm{Ric}(V, W) - \frac{1}{2}R\langle V, W\rangle$$

$$= \frac{1}{2}\Big(\langle R_{\underline{L}V}L, W\rangle + \langle R_{LV}\underline{L}, W\rangle\Big) + \mathrm{tr}_{\gamma}\langle R_{(\cdot)V}(\cdot), W\rangle$$

$$+ \frac{1}{2}(G(\underline{L}, L) + \mathrm{tr}_{\gamma}\, G)\langle V, W\rangle.$$

Since Σ is of dimension two we must have that $\mathrm{tr}_{\gamma}\langle R_{(\cdot)V}(\cdot), W\rangle \propto \langle V, W\rangle$ with factor of proportionality $\mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H}\rangle + \frac{1}{2}\underline{\hat{\chi}} \cdot \hat{\chi}$ coming from Proposition 3.0.1. We conclude therefore that the symmetric part satisfies

$$\frac{1}{2}\Big(\langle R_{V\underline{L}}W, L\rangle + \langle R_{W\underline{L}}V, L\rangle\Big) = \hat{G}(V, W) - (\mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H}\rangle + \frac{1}{2}\underline{\hat{\chi}}\cdot\hat{\chi} + \frac{1}{2}G(\underline{L}, L))\langle V, W\rangle$$

and the result follows as soon as we sum up the antisymmetric and symmetric contributions. □

Combining the previous lemma with the propagation of χ we have

$$
\begin{aligned}
(\mathcal{L}_{\underline{L}}\chi)(V,W) &= \Big(\mathcal{K} - \frac{1}{4}\langle \vec{H},\vec{H}\rangle + \frac{1}{2}\hat{\underline{\chi}}\cdot\hat{\chi} + \frac{1}{2}G(\underline{L},L)\Big)\langle V,W\rangle - \hat{G}(V,W) \\
&\quad + (\mathrm{curl}\zeta)(V,W) - \frac{1}{2}\Big(\langle \vec{\chi}(V),\vec{\underline{\chi}}(W)\rangle - \langle \vec{\underline{\chi}}(V),\vec{\chi}(W)\rangle\Big) \\
&\quad - 2(\nabla\!\!\!/_V\zeta)(W) - \kappa\chi(V,W) + \langle \vec{\chi}(V),\vec{\underline{\chi}}(W)\rangle - 2\zeta(V)\zeta(W) \\
&= \Big(\mathcal{K} - \frac{1}{4}\langle \vec{H},\vec{H}\rangle + \frac{1}{2}\hat{\underline{\chi}}\cdot\hat{\chi} + \frac{1}{2}G(\underline{L},L)\Big)\langle V,W\rangle - \hat{G}(V,W) \\
&\quad + \frac{1}{2}\Big(\langle \vec{\underline{\chi}}(V),\vec{\chi}(W)\rangle + \langle \vec{\chi}(V),\vec{\underline{\chi}}(W)\rangle\Big) - (\nabla\!\!\!/_V\zeta)(W) - (\nabla\!\!\!/_W\zeta)(V) \\
&\quad - 2\zeta(V)\zeta(W) - \kappa\chi(V,W).
\end{aligned}
$$

Using again the fact that $AB + BA = \mathrm{tr}(AB)\mathbb{I}$ for symmetric, traceless 2×2 matrices it follows that

$$
\begin{aligned}
\langle \vec{\chi}(V),\vec{\underline{\chi}}(W)\rangle + \langle \vec{\underline{\chi}}(V),\vec{\chi}(W)\rangle &= (\hat{\chi}\cdot\hat{\underline{\chi}})\langle V,W\rangle + \mathrm{tr}\,\chi\hat{\underline{\chi}}(V,W) + \mathrm{tr}\,\underline{\chi}\hat{\chi}(V,W) \\
&\quad + \frac{1}{2}\langle \vec{H},\vec{H}\rangle\langle V,W\rangle
\end{aligned}
$$

giving the result.

6.
$$
\begin{aligned}
\underline{L}\,\mathrm{tr}\,\chi &= \mathcal{L}_{\underline{L}}(C_1^1C_2^2\gamma^{-1}\otimes\chi) \\
&= -2\underline{\chi}\cdot\chi + \mathrm{tr}(\mathcal{L}_{\underline{L}}\chi) \\
&= -2\underline{\chi}\cdot\chi + 2\mathcal{K} + 2\hat{\underline{\chi}}\cdot\hat{\chi} + G(\underline{L},L) - 2\nabla\!\!\!/\cdot\zeta - 2|\zeta|^2 - \kappa\,\mathrm{tr}\,\chi \\
&= G(\underline{L},L) + 2\mathcal{K} - 2\nabla\!\!\!/\cdot\zeta - 2|\zeta|^2 - \langle \vec{H},\vec{H}\rangle - \kappa\,\mathrm{tr}\,\chi
\end{aligned}
$$

7. And finally,

$$
\begin{aligned}
(\mathcal{L}_{\underline{L}}\zeta)(V) = \underline{L}\zeta(V) &= \frac{1}{2}\underline{L}\langle D_V\underline{L},L\rangle \\
&= \frac{1}{2}\langle D_{\underline{L}}D_V\underline{L},L\rangle + \frac{1}{2}\langle D_V\underline{L},D_{\underline{L}}L\rangle \\
&= \frac{1}{2}\langle R_{V\underline{L}}\underline{L} + D_VD_{\underline{L}}\underline{L},L\rangle + \frac{1}{2}\langle \vec{\underline{\chi}}(V) + \zeta(V)\underline{L}, -2\vec{\zeta} - \kappa L\rangle
\end{aligned}
$$

$$= \frac{1}{2}\langle R_{V\underline{L}}\underline{L}, L\rangle + V\kappa + \kappa\zeta(V) - \underline{\chi}(V, \vec{\zeta}) - \kappa\zeta(V)$$

$$= -\nabla\!\!\!/ \cdot \hat{\underline{\chi}}(V) + \frac{1}{2}V \operatorname{tr}\underline{\chi} - \operatorname{tr}\underline{\chi}\zeta(V) + G(V, \underline{L}) + V\kappa$$

having used Lemma 4.1.1 to obtain the third line and the Codazzi equation (3.2) to get the fifth.

□

4.3 Propagation of ρ

From Proposition 4.2.1 we have the propagation of the first two terms of ρ for the third and forth we'll need

Corollary 4.3.0.1. *Assuming $\{\Sigma_s\}$ is expanding along $\underline{L}$ we have*

$$\underline{L}(\nabla\!\!\!/ \cdot \zeta) = -2\nabla\!\!\!/ \cdot (\hat{\underline{\chi}} \cdot \zeta) - 2\operatorname{tr}\underline{\chi}\nabla\!\!\!/ \cdot \zeta - \nabla\!\!\!/ \cdot \nabla\!\!\!/ \cdot \hat{\underline{\chi}} + \frac{1}{2}\Delta\!\!\!/ \operatorname{tr}\underline{\chi} - d\!\!\!/ \operatorname{tr}\underline{\chi} \cdot \zeta$$
$$+ \nabla\!\!\!/ \cdot G_{\underline{L}} + \Delta\!\!\!/\kappa$$

$$\underline{L}\Delta\!\!\!/ \log\operatorname{tr}\underline{\chi} = -2\nabla\!\!\!/ \cdot (\hat{\underline{\chi}} \cdot d\!\!\!/ \log\operatorname{tr}\underline{\chi}) - \frac{3}{2}\operatorname{tr}\underline{\chi}\Delta\!\!\!/ \log\operatorname{tr}\underline{\chi} - \frac{1}{2}\operatorname{tr}\underline{\chi}|d\!\!\!/ \log\operatorname{tr}\underline{\chi}|^2$$
$$- \Delta\!\!\!/ \frac{|\hat{\underline{\chi}}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr}\underline{\chi}} + \Delta\!\!\!/\kappa$$

Proof. When used we will assume $V, W \in E(\Sigma)$.

For any 1-form η on Ω we have

$$\mathcal{L}_{\underline{L}}(\nabla\!\!\!/\eta)(V, W) = \underline{L}(\nabla\!\!\!/_V\eta(W)) = V\underline{L}\eta(W) - \underline{L}\eta(\nabla\!\!\!/_V W)$$
$$= V(\mathcal{L}_{\underline{L}}\eta)(W) - (\mathcal{L}_{\underline{L}}\eta)(\nabla\!\!\!/_V W) - \eta([\underline{L}, \nabla\!\!\!/_V W])$$
$$= \nabla\!\!\!/_V(\mathcal{L}_{\underline{L}}\eta)(W) - \eta([\underline{L}, \nabla\!\!\!/_V W])$$

from which we find

$$\underline{L}(\nabla\!\!\!/ \cdot \eta) = C_1^1 C_2^2(\mathcal{L}_{\underline{L}}\gamma^{-1} \otimes \nabla\!\!\!/\eta + \gamma^{-1} \otimes \mathcal{L}_{\underline{L}}(\nabla\!\!\!/\eta))$$

$$= -2\underline{\chi} \cdot \nabla\!\!\!/ \eta + \operatorname{tr}(\mathcal{L}_{\underline{L}} \nabla\!\!\!/ \eta)$$

$$= -2(\hat{\underline{\chi}} + \frac{1}{2}\operatorname{tr}\underline{\chi}\gamma) \cdot \nabla\!\!\!/ \eta + \nabla\!\!\!/ \cdot (\mathcal{L}_{\underline{L}}\eta) - \eta(2\vec{\nabla\!\!\!/} \cdot \hat{\underline{\chi}})$$

the last term coming from Lemma 4.2.1 after taking a trace over V, W. We conclude that

$$\underline{L}(\nabla\!\!\!/ \cdot \eta) = -\operatorname{tr}\underline{\chi} \nabla\!\!\!/ \cdot \eta - 2\nabla\!\!\!/ \cdot (\hat{\underline{\chi}} \cdot \eta) + \nabla\!\!\!/ \cdot (\mathcal{L}_{\underline{L}}\eta).$$

The first part of the corollary now straight forwardly follows from Proposition 4.2.1 for $\eta = \zeta$. For the second, since $\Delta\!\!\!/ \log \operatorname{tr}\underline{\chi} = \nabla\!\!\!/ \cdot d\!\!\!/ \log \operatorname{tr}\underline{\chi}$ we have

$$\underline{L}\Delta\!\!\!/ \log \operatorname{tr}\underline{\chi} = -\operatorname{tr}\underline{\chi}\Delta\!\!\!/ \log \operatorname{tr}\underline{\chi} - 2\nabla\!\!\!/ \cdot (\hat{\underline{\chi}} \cdot d\!\!\!/ \log \operatorname{tr}\underline{\chi}) + \nabla\!\!\!/ \cdot (\mathcal{L}_{\underline{L}} d\!\!\!/ \log \operatorname{tr}\underline{\chi}).$$

From the fact that

$$\nabla\!\!\!/ \cdot (\mathcal{L}_{\underline{L}} d\!\!\!/ \log \operatorname{tr}\underline{\chi}) = \nabla\!\!\!/ \cdot (d\!\!\!/ \underline{L} \log \operatorname{tr}\underline{\chi}) = \Delta\!\!\!/ \Big(-\frac{1}{2}\operatorname{tr}\underline{\chi} - \frac{|\hat{\underline{\chi}}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr}\underline{\chi}} + \kappa \Big)$$

the result follows as soon as we make the substitution

$$\Delta\!\!\!/ \operatorname{tr}\underline{\chi} = \operatorname{tr}\underline{\chi}\Big(\Delta\!\!\!/ \log \operatorname{tr}\underline{\chi} + |d\!\!\!/ \log \operatorname{tr}\underline{\chi}|^2\Big)$$

□

Theorem 4.3.1 (Propagation of ρ)**.** *Assuming $\{\Sigma_s\}$ is expanding along the flow vector $\underline{L} = \sigma L^-$ we conclude that*

$$\dot{\rho} + \frac{3}{2}\sigma\rho = \frac{\sigma}{2}\Big(\frac{1}{2}\langle \vec{H}, \vec{H} \rangle \Big(|\hat{\chi}^-|^2 + G(L^-, L^-)\Big) + |\tau|^2 - \frac{1}{2}G(L^-, L^+)\Big)$$
$$+ \Delta\!\!\!/ \Big(\sigma(|\hat{\chi}^-|^2 + G(L^-, L^-))\Big) - 2\nabla\!\!\!/ \cdot (\sigma\hat{\chi}^- \cdot \tau) + \nabla\!\!\!/ \cdot (\sigma G_{L^-})$$

Proof. From Proposition 4.2.1 and Corollary 4.3.0.1 the proof reduces to an exercise in algebraic manipulation

$$\mathcal{L}_{\underline{L}}\rho = \mathcal{L}_{\underline{L}}K - \frac{1}{4}tr\chi\mathcal{L}_{\underline{L}}\operatorname{tr}\underline{\chi} - \frac{1}{4}\operatorname{tr}\underline{\chi}\mathcal{L}_{\underline{L}}tr\chi + \underline{L}\nabla\!\!\!/ \cdot \zeta - \underline{L}\Delta\!\!\!/ \log \operatorname{tr}\underline{\chi}$$

$$= \Big(\nabla\!\!\!/ \cdot \nabla\!\!\!/ \cdot \underline{\hat{\chi}} - \frac{1}{2}\Delta\!\!\!/ \operatorname{tr}\underline{\chi} - \operatorname{tr}\underline{\chi}K\Big) - \frac{1}{4}\operatorname{tr}\chi\Big(- \frac{1}{2}\operatorname{tr}^2\underline{\chi} - |\underline{\hat{\chi}}|^2 - G(\underline{L},\underline{L}) + \kappa\operatorname{tr}\underline{\chi}\Big)$$

$$- \frac{1}{4}\operatorname{tr}\underline{\chi}\Big(G(\underline{L},L) + 2K - 2\nabla\!\!\!/ \cdot \zeta - 2|\zeta|^2 - \langle\vec{H},\vec{H}\rangle - \kappa\operatorname{tr}\chi\Big)$$

$$- 2\nabla\!\!\!/ \cdot (\underline{\hat{\chi}} \cdot \zeta) - 2\operatorname{tr}\underline{\chi}\nabla\!\!\!/ \cdot \zeta - \nabla\!\!\!/ \cdot \nabla\!\!\!/ \cdot \underline{\hat{\chi}} + \frac{1}{2}\Delta\!\!\!/ \operatorname{tr}\underline{\chi} - d\!\!\!/ \operatorname{tr}\underline{\chi} \cdot \zeta + \nabla\!\!\!/ \cdot G_{\underline{L}} + \Delta\!\!\!/ \kappa$$

$$+ 2\nabla\!\!\!/ \cdot (\underline{\hat{\chi}} \cdot d\!\!\!/ \log\operatorname{tr}\underline{\chi}) + \frac{3}{2}\operatorname{tr}\underline{\chi}\Delta\!\!\!/ \log\operatorname{tr}\underline{\chi} + \frac{1}{2}\operatorname{tr}\underline{\chi}|d\!\!\!/ \log\operatorname{tr}\underline{\chi}|^2$$

$$+ \Delta\!\!\!/ \frac{|\underline{\hat{\chi}}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}\underline{\chi}} - \Delta\!\!\!/ \kappa$$

$$= -\frac{3}{2}\operatorname{tr}\underline{\chi}K + \frac{1}{8}\operatorname{tr}\underline{\chi}\langle\vec{H},\vec{H}\rangle + \frac{1}{4}\langle\vec{H},\vec{H}\rangle\Big(\frac{|\underline{\hat{\chi}}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}\underline{\chi}}\Big) - \frac{1}{4}\operatorname{tr}\underline{\chi}G(\underline{L},L)$$

$$- \frac{3}{2}\operatorname{tr}\underline{\chi}\nabla\!\!\!/ \cdot \zeta + \frac{1}{4}\operatorname{tr}\underline{\chi}\langle\vec{H},\vec{H}\rangle - 2\nabla\!\!\!/ \cdot (\underline{\hat{\chi}} \cdot (\zeta - d\!\!\!/ \log\operatorname{tr}\underline{\chi})) + \frac{3}{2}\operatorname{tr}\underline{\chi}\Delta\!\!\!/ \log\operatorname{tr}\underline{\chi}$$

$$+ \frac{1}{2}\operatorname{tr}\underline{\chi}|\zeta|^2 - d\!\!\!/ \operatorname{tr}\underline{\chi} \cdot \zeta + \frac{1}{2}\operatorname{tr}\underline{\chi}|d\!\!\!/ \log\operatorname{tr}\underline{\chi}|^2$$

$$= -\frac{3}{2}\operatorname{tr}\underline{\chi}\rho + \frac{1}{4}\langle\vec{H},\vec{H}\rangle\Big(\frac{|\underline{\hat{\chi}}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}\underline{\chi}}\Big) + \frac{1}{2}\operatorname{tr}\underline{\chi}|\zeta - d\!\!\!/ \log\operatorname{tr}\underline{\chi}|^2 - \frac{1}{4}\operatorname{tr}\underline{\chi}G(\underline{L},L)$$

$$+ \Delta\!\!\!/ \frac{|\underline{\hat{\chi}}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}\underline{\chi}} - 2\nabla\!\!\!/ \cdot (\underline{\hat{\chi}} \cdot (\zeta - d\!\!\!/ \log\operatorname{tr}\underline{\chi})) + \nabla\!\!\!/ \cdot G_{\underline{L}}.$$

The result therefore follows as soon as we express all terms according to the inflation basis $\{L^-, L^+\}$ where $\{\Sigma_s\}$ is a flow along $\underline{L} = \sigma L^-$ of speed $\sigma = \operatorname{tr}\underline{\chi}$. □

Corollary 4.3.1.1. *For $\{\Sigma_s\}$ expanding along the flow vector $\underline{L} = \sigma L^-$ and any $u \in \mathcal{F}(\Sigma_s)$*

$$\int_{\Sigma_s} e^u\Big(\dot{\rho} + \frac{3}{2}\sigma\rho\Big)dA = \int_{\Sigma_s} \sigma e^u\Big(\Big(|\hat{\chi}^-|^2 + G(L^-,L^-)\Big)\Big(\frac{1}{4}\langle\vec{H},\vec{H}\rangle + \Delta\!\!\!/ u\Big)$$

$$+ \frac{1}{2}|2\hat{\chi}^- \cdot d\!\!\!/ u + \tau|^2 + G(L^-, |\nabla\!\!\!/ u|^2L^- - \nabla\!\!\!/ u - \frac{1}{4}L^+)\Big)dA$$

Proof. We start by integrating by parts on the last three terms of Theorem 4.3.1

$$\int e^u\Big(\Delta\!\!\!/ (\sigma(|\hat{\chi}^-|^2 + G(L^-,L^-))) - 2\nabla\!\!\!/ \cdot (\sigma\hat{\chi}^- \cdot \tau) + \nabla\!\!\!/ \cdot (\sigma G_{L^-})\Big)dA$$

$$= \int \sigma e^u \Big(e^{-u}(\Delta\!\!\!/ e^u)(|\hat\chi^-|^2 + G(L^-, L^-)) + 2\hat\chi^-(\nabla\!\!\!/ u, \vec\tau) - G(L^-, \nabla\!\!\!/ u)\Big) dA$$

$$= \int \sigma e^u \Big((\Delta\!\!\!/ u + |\nabla\!\!\!/ u|^2)(|\hat\chi^-|^2 + G(L^-, L^-)) + 2\hat\chi^-(\nabla\!\!\!/ u, \vec\tau) - G(L^-, \nabla\!\!\!/ u)\Big) dA$$

$$= \int \sigma e^u \Big((|\hat\chi^-|^2 + G(L^-, L^-))\Delta\!\!\!/ u + |\hat\chi^-|^2 |\nabla\!\!\!/ u|^2 + 2\underline{\hat\chi}(\nabla\!\!\!/ u, \vec\tau)$$

$$+ G(L^-, |\nabla\!\!\!/ u|^2 L^- - \nabla\!\!\!/ u)\Big) dA.$$

As a result

$$\int e^u \Big(\dot\rho + \frac{3}{2}\sigma\rho\Big) dA = \int \sigma e^u \Big((|\hat\chi^-|^2 + G(L^-, L^-))\Big(\frac{1}{4}\langle \vec H, \vec H\rangle + \Delta\!\!\!/ u\Big)$$

$$+ |\hat\chi^-|^2 |\nabla\!\!\!/ u|^2 + 2\hat\chi^-(\nabla\!\!\!/ u, \vec\tau) + \frac{1}{2}|\tau|^2 + G(L^-, |\nabla\!\!\!/ u|^2 L^- - \nabla\!\!\!/ u - \frac{1}{4}L^+)\Big) dA.$$

Since $\hat\chi^-$ is symmetric and trace-free it follows that $|\hat\chi^- \cdot d\!\!\!/ u|^2 = \frac{1}{2}|\hat\chi^-|^2 |\nabla\!\!\!/ u|^2$ from which the first three terms of the second line simplifies to give

$$|\hat\chi^-|^2 |\nabla\!\!\!/ u|^2 + 2\hat\chi^-(\nabla\!\!\!/ u, \vec\tau) + \frac{1}{2}|\tau|^2 = \frac{1}{2}|2\hat\chi^- \cdot d\!\!\!/ u + \tau|^2$$

□

Remark 4.3.1. *An interesting consequence of the above corollary in spacetimes satisfying the dominant energy condition is the fact that any $u \in \mathcal{F}(\Sigma)$ gives*

$$\int e^u \Big(\dot\rho + \frac{3}{2}\sigma\rho\Big) dA \geqslant \int \sigma e^u (|\hat\chi^-|^2 + G(L^-, L^-))\Big(\frac{1}{4}\langle \vec H, \vec H\rangle + \Delta\!\!\!/ u\Big) dA$$

The proof of Theorem 2.1.1 is a simple consequence of the following corollary:

Corollary 4.3.1.2. *Assuming $\{\Sigma_s\}$ is expanding along the flow vector $\underline{L} = \sigma L^-$ with each Σ_s of non-zero flux ($|\rho(s)| > 0$) then*

$$\frac{d}{ds}\int_{\Sigma_s} \rho^{\frac{2}{3}} dA = \int_{\Sigma_s} (\dot{\rho^{\frac{2}{3}}}) + \sigma \rho^{\frac{2}{3}} dA$$

$$= \frac{2}{3}\int_\Sigma \frac{\sigma}{\rho^{\frac{1}{3}}}\Big(\Big(|\hat{\chi}^-|^2 + G(L^-,L^-)\Big)\Big(\frac{1}{4}\langle\vec{H},\vec{H}\rangle - \frac{1}{3}\not\Delta\log|\rho|\Big)$$

$$+ \frac{1}{2}|\frac{2}{3}\hat{\chi}^- \cdot \not d\log|\rho| - \tau|^2$$

$$+ G(L^-, \frac{1}{9}|\not\nabla\log|\rho||^2 L^- + \frac{1}{3}\not\nabla\log|\rho| - \frac{1}{4}L^+)\Big)dA$$

Proof. From the first variation of Area formula

$$\dot{dA} = -\langle\vec{H},\underline{L}\rangle dA = -\sigma\langle\vec{H},L^-\rangle dA = \sigma dA$$

we get the first equality. For the second we apply Corollary 4.3.1.1 with $e^u = \frac{2}{3}|\rho|^{-\frac{1}{3}}$, canceling the sign in the case that $\rho < 0$. □

4.3.1 Case of Equality

Lemma 4.3.1. *For $\{\Sigma_s\}$ expanding along $\underline{L} = \sigma L^-$ we have*

$$\mathcal{L}_{\underline{L}}\tau + \sigma\tau + \not\nabla\cdot(\sigma\hat{\chi}^-) = \sigma G_{L^-} + \not d(\sigma(|\hat{\chi}^-|^2 + G(L^-,L^-)))$$

Proof. By combining (8) and (11):

$$\mathcal{L}_{\underline{L}}(\zeta - \not d\log\operatorname{tr}\underline{\chi}) = G_{\underline{L}} - \not\nabla\cdot\hat{\underline{\chi}} - \operatorname{tr}\underline{\chi}\zeta + \frac{1}{2}\not d\operatorname{tr}\underline{\chi} + \not d\kappa$$

$$- \not d\Big(-\frac{1}{2}\operatorname{tr}\underline{\chi} - \frac{|\hat{\underline{\chi}}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}\underline{\chi}} + \kappa\Big)$$

$$= -\operatorname{tr}\underline{\chi}(\zeta - \not d\log\operatorname{tr}\underline{\chi}) - \not\nabla\cdot\hat{\underline{\chi}} + G_{\underline{L}} + \not d\frac{|\hat{\underline{\chi}}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}\underline{\chi}}.$$

The result follows as soon as we switch to the inflation basis $\{L^-, L^+\}$. □

Theorem 4.3.2. *Let Ω be a null hypersurface in a spacetime satisfying the dominant energy condition with vector field $\underline{L}$ tangent to the null generators of Ω. Suppose $\{\Sigma_s\}$ is an expanding (SP)-foliation defined as the level sets of a function $s : \Omega \to \mathbb{R}$ satisfying $\underline{L}(s) = 1$ and achieves the case of equality $\frac{dm}{ds} = 0$. Then all foliations achieve* *equality, moreover, we find an affine level set function* $r \in \mathcal{F}(\Omega)$ *with* $r_0 := r|_{\Sigma_{s_0}} \circ \pi$ *such that any surface* $\Sigma := \{r = \omega \circ \pi\}$*, for* $\omega \in \mathcal{F}(\Sigma_{s_0})$*, has data:*

$$\begin{aligned}
\gamma &= \omega^2 \gamma_0 \\
\underline{\chi} &= \omega \gamma_0 \\
\operatorname{tr} \underline{\chi} &= \frac{2}{\omega} \\
\operatorname{tr} \chi &= \frac{2}{\omega}(\mathcal{K}_0 - \frac{r_0}{\omega} - \omega^2 \not{\Delta} \log \omega) \\
\zeta &= -\not{d} \log \omega \\
\rho &= \frac{r_0}{\omega^3}
\end{aligned}$$

where $r_0^2 \gamma_0$ *is the metric on* Σ_{s_0} *and* $\mathcal{K}_0$ *the Gaussian curvature associated to* γ_0*. In the case that* $\operatorname{tr} \chi|_{\Sigma_{s_0}} = 0$ *our data corresponds with the the standard nullcone in Schwarzschild spacetime of mass* $M = \frac{r_0}{2}$*.*

Proof. Without loss of generality we assume $s_0 = 0$. Immediately from Corollary 4.3.1.2 we conclude for this particular foliation that

$$\begin{aligned}
|\hat{\chi}^-|^2 + G(L^-, L^-) &= 0 \\
|\frac{2}{3}\hat{\chi}^- \cdot \not{d} \log \rho - \tau|^2 &= 0 \\
G(L^-, \frac{1}{9}|\not{\nabla} \log \rho|^2 L^- + \frac{1}{3}\not{\nabla} \log \rho - \frac{1}{4}L^+) &= 0.
\end{aligned}$$

So from the first equality we have both $\hat{\chi}^- = 0$ and $G(L^-, L^-) = 0$. Combined with the second equality we conclude that $\tau = 0$ for this particular foliation and therefore Lemma 4.3.1 ensures that $G_{L^-} = 0$ as well. Finally we may therefore utilize the final equality to conclude also that $G(L^+, L^-) = 0$ so that, for any $p \in \Omega$ and any $X \in T_pM$, we have

$$G(L^-, X) = 0.$$

From this and Lemma 4.3.1 we have for any foliation off of Σ_0 generated by some $\underline{L}_a$ $(a > 0)$ that

$$\mathcal{L}_{\underline{L}_a}\tau^a + a\sigma\tau^a = 0.$$

Given that $\tau^a|_{\Sigma_0} = \tau|_{\Sigma_0} = 0$ this enforces $\tau^a = 0$ by standard uniqueness theorems. We recognise this implies the case of equality for all foliations so without loss of generality we assume that $\underline{L}$ is geodesic. We are now in a position to show that the flux $\rho \in \mathcal{F}(\Omega)$ is independent of the foliation from which it is constructed. In particular, for any $a > 0$, foliating off of Σ_0 along the generator $\underline{L}_a$ will construct a ρ_a which we would like to show agrees pointwise on Ω with ρ.

From Theorem 4.3.1 we have

$$\underline{L}\rho = -\frac{3}{2}\operatorname{tr}\underline{\chi}\rho = 3\rho\underline{L}\log\operatorname{tr}\underline{\chi}$$

so for any $p \in \Omega$ solving this ODE along the geodesic $\gamma^{\underline{L}}_{\pi(p)}(s)$ gives

$$\frac{\rho \circ s(p)}{\rho(0)} = \Big(\frac{\operatorname{tr}\underline{\chi}(p)}{\operatorname{tr}\underline{\chi}(0)}\Big)^3.$$

For the generator $\underline{L}_a$ Theorem 4.3.1 gives

$$\begin{aligned}\underline{L}_a\rho_a = -\frac{3}{2}\operatorname{tr}\underline{\chi}_a\rho_a &= 3\rho_a(\underline{L}_a\log\operatorname{tr}\underline{\chi}_a - \kappa_a)\\ &= 3\rho_a\underline{L}_a(\log\operatorname{tr}\underline{\chi}_a - \log a)\\ &= 3\rho_a\underline{L}_a(\log\operatorname{tr}\underline{\chi})\end{aligned}$$

where the penultimate line comes from the fact that

$$\kappa_a\underline{L}_a = D_{\underline{L}_a}\underline{L}_a = a\underline{L}(a)\underline{L} = \underline{L}_a(\log a)\underline{L}_a$$

and the final line from the fact that $\operatorname{tr}\underline{\chi}_a = a\operatorname{tr}\underline{\chi}$. Solving this ODE along the pregeodesic $\gamma^{\underline{L}_a}_{\pi(p)}(t)$ we have

$$\frac{\rho_a \circ t(p)}{\rho_a(0)} = \Big(\frac{\operatorname{tr}\underline{\chi}(p)}{\operatorname{tr}\underline{\chi}(0)}\Big)^3 = \frac{\rho \circ s(p)}{\rho(0)}.$$

Since we're foliating off of Σ_0 in both cases and $\rho|_{\Sigma_0}$ is independent of our choice of null basis we have $\rho(p) = \rho_a(p)$ as desired.

We therefore define the functions r_0 and r according to

$$\frac{1}{r_0^2} = \rho|_{\Sigma_{s_0}}, \quad \frac{r_0 \circ \pi}{r^3} = \rho$$

(i.e. $r|_{\Sigma_0} = r_0$) so that Theorem 4.3.1 gives

$$-3\frac{r_0}{r^4}\underline{L}_a(r) = \underline{L}_a(\rho) = -\frac{3}{2}\operatorname{tr}\underline{\chi}_a\rho = -\frac{3}{2}\operatorname{tr}\underline{\chi}_a\frac{r_0}{r^3}$$

and therefore $\underline{L}_a(r) = \frac{1}{2}\operatorname{tr}\underline{\chi}_a r$. It follows that if we scale $\underline{L}$ such that $\operatorname{tr}\underline{\chi}|_{\Sigma_0} = \frac{2}{r_0}$ then $\underline{L}(\operatorname{tr}\underline{\chi}r) = -\frac{1}{2}(\operatorname{tr}\underline{\chi})^2 r + \operatorname{tr}\underline{\chi}(\frac{1}{2}\operatorname{tr}\underline{\chi}r) = 0$ implies that $\operatorname{tr}\underline{\chi} = \frac{2}{r}$ and $\underline{L}(r) = 1$. So r is in fact our level set function. For $r_0^2\gamma_0$ the metric on Σ_0, by Lie dragging γ_0 along $\underline{L}$ to all of Ω we have

$$\mathcal{L}_{\underline{L}}(r^2\gamma_0) = 2r\gamma_0 = \frac{2}{r}(r^2\gamma_0) = \operatorname{tr}\underline{\chi}(r^2\gamma_0).$$

So from (6), $\mathcal{L}_{\underline{L}}(r^2\gamma_0 - \gamma) = \operatorname{tr}\underline{\chi}(r^2\gamma_0 - \gamma)$ and $r_0^2\gamma_0 - \gamma(r_0) = 0$ giving $\gamma(r) = r^2\gamma_0$ by uniqueness. We conclude that for any $0 \leqslant \omega \in \mathcal{F}(\Sigma_0)$ the cross-section $\Sigma := \{r = \omega \circ \pi\}$ has metric $\gamma_\omega = \gamma(r)|_\Sigma = \omega^2\gamma_0$ with Gaussian curvature $\mathcal{K}_\omega = \frac{1}{\omega^2}\mathcal{K}_0 - \not{\Delta}\log\omega$. Moreover,

$$\begin{aligned}\frac{r_0}{\omega^3} = \rho_\omega &= \mathcal{K}_\omega - \frac{1}{4}\langle\vec{H},\vec{H}\rangle \\ &= \frac{1}{\omega^2}\mathcal{K}_0 - \not{\Delta}\log\omega - \frac{1}{2\omega}\operatorname{tr}\chi_\omega\end{aligned}$$

having used the fact that $\rho_\omega = \rho|_\Sigma$ (from independence of foliation) in the first line and $\operatorname{tr}\underline{\chi}_\omega = \operatorname{tr}\underline{\chi}|_\Sigma$ in the second. We conclude that,

$$\operatorname{tr}\chi_\omega = \frac{2}{\omega}(\mathcal{K}_0 - \frac{r_0}{\omega} - \omega^2\not{\Delta}\log\omega).$$

In the case that $\operatorname{tr}\chi|_{\Sigma_0} = 0$ property (SP) forces $\frac{1}{r_0^2} = \rho|_{\Sigma_0}$ to be constant by way of the maximum principle. From our expression for $\operatorname{tr}\chi_{r_0}$ we conclude that $\mathcal{K}_0 = 1$ and therefore γ_0 is a round metric on $\mathbb{S}^2$. □

Remark 4.3.2. *We bring to the attention of the reader that due the lack of information regarding the term $\hat{G}$ in (4.5) we are unable to conclude with any knowledge of the datum χ on Σ. In the case of vacuum this no longer poses a problem and one is able to correlate $\chi|_{\Sigma}$ with $\chi|_{\Sigma_{r_0}}$ as shown by Sauter ([25], Lemma 4.3).*

5

Foliation Comparison

In this chapter we show how the flux function ρ of an arbitrary cross-section of Ω decomposes in terms of the flux of the background foliation. With the appropriate asymptotic decay on Ω this allows us to prove Theorem 2.1.2.

5.1 Additional Setup

We follow once again the construction of [18] starting with a background foliation as constructed in Section 4.1 off of an initial cross-section Σ_{s_0}. As before, each Σ_s allows a null basis $\{\underline{L}, l\}$ such that $\langle \underline{L}, l \rangle = 2$. Also from Section 4.1 we have the diffeomorphism $p \mapsto (\pi(p), s(p))$ of Ω onto its image. Therefore any cross-section with associated embedding $\Phi : \mathbb{S}^2 \to \Omega$ is equivalently realized with the map $\tilde{\Phi} = (\pi, s) \circ \Phi$. Expressing the component functions $\Psi := \pi \circ \Phi$ and $\omega := s \circ \Phi$ we recognize that $\Psi : \mathbb{S}^2 \to \Sigma_{s_0}$ is a diffeomorphism and therefore the embedding $\Phi : \mathbb{S}^2 \to \Omega$ is uniquely characterized as a graph over Σ_{s_0} with graph function $\omega \circ \Psi^{-1}$. Without confusion we will simply denote the graph function by ω and it's associated cross-section by Σ_ω. We wish to compare both the intrinsic and extrinsic geometry of Σ_ω at a point q with the geometry of the surface $\Sigma_{s(q)}$. We extend ω to all of Ω in the usual way by

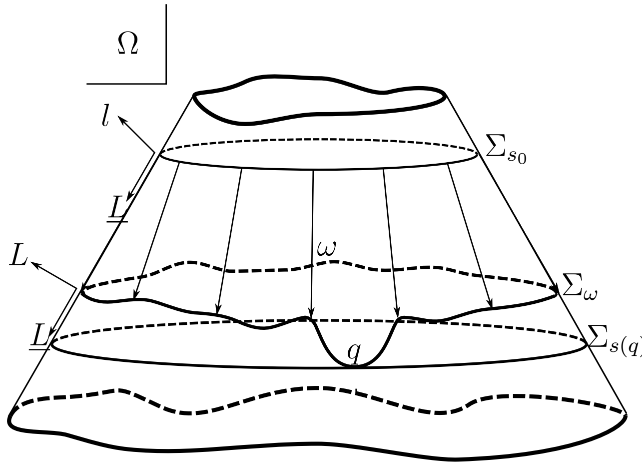

FIGURE 5.1: Σ_ω as a graph over Σ_{s_0}

imposing it be constant along generators of $\underline{L}$, in other words, $\omega(p) := (\omega \circ \pi)(p)$. For the extrinsic geometry of Σ_ω we have the null-normal basis $\{\underline{L}, L\}$ whereby L is given by the conditions $\langle \underline{L}, L\rangle = 2$ and $\langle V, L\rangle = 0$ for any $V \in \Gamma(T\Sigma_\omega)$. As before Σ_ω has second fundamental form decomposing into the null components $\underline{\chi}$ (associated to $\underline{L}$) and χ (associated to L) with torsion ζ. For each Σ_s we equivalently decompose the second fundamental form into the components K (associated to $\underline{L}$) and Q (associated to l) with torsion t. We will denote the induced covariant derivative on Σ_s by ∇ and on Σ_ω by $\not\nabla$. The following lemma is known ([18],[25]):

Lemma 5.1.1. *Given $q \in \Sigma_\omega \cap \Sigma_{s(q)}$ the map given by*

$$T_\omega : T_q\Sigma_{s(q)} \to T_q\Sigma_\omega$$
$$v \to \tilde{v} := v + v\omega\underline{L}$$

is a well defined isomorphism with natural extension $E(\Sigma_{s_0}) \to E(\Sigma_\omega)$. Moreover,

- $\gamma_\omega(\tilde{V}, \tilde{W}) = \gamma_s(V, W)$
- $\underline{\chi}(\tilde{V}, \tilde{W}) = K(V, W)$
- $\zeta(\tilde{V}) = t(V) - K(V, \nabla\omega) + \kappa\langle V, \nabla\omega\rangle$

- $\chi(\tilde{V}, \tilde{W}) = Q(V, W) - 2t(V)\langle W, \nabla\omega\rangle - 2t(W)\langle V, \nabla\omega\rangle - |\nabla\omega|^2 K(V, W)$
$$- 2H^\omega(V, W) + 2K(V, \nabla\omega)\langle W, \nabla\omega\rangle + 2K(W, \nabla\omega)\langle V, \nabla\omega\rangle$$
$$- 2\kappa\langle V, \nabla\omega\rangle\langle W, \nabla\omega\rangle$$

- $\operatorname{tr}\chi = \operatorname{tr}Q - 4t(\nabla\omega) - 2(\Delta\omega - 2\hat{K}(\nabla\omega, \nabla\omega)) + \operatorname{tr}K|\nabla\omega|^2 - 2\kappa|\nabla\omega|^2$

for H^ω the Hessian of ω on Σ_s.

Proof. For completeness we include a similar proof as in [18] (Proposition 1). Since $T_\omega : T_q\Sigma_{s(q)} \to T_q\Sigma_\omega$ is clearly injective it suffices to show $\tilde{v} \in T_q\Sigma_\omega$. This follows from the fact that $\tilde{v}(s - \omega) = v(s - \omega) + v\omega\underline{L}(s - \omega) = -v\omega + v\omega = 0$ since Σ_ω is locally characterised by $s|_{\Sigma_\omega} = \omega$. For the extension $\tilde{V} = V + V\omega\underline{L}$ we note that $[\underline{L}, V] = 0 \implies [\underline{L}, \tilde{V}] = 0$ and it follows that $\tilde{V} \in E(\Sigma_\omega)$ (infact $\tilde{E}(\Sigma_{s_0}) = E(\Sigma_\omega)$). From this and the fact that $D_{\underline{L}}\underline{L} = \kappa\underline{L}$ the first two identities follow straight forwardly. For the third identity we find that $L = l - |\nabla\omega|^2\underline{L} - 2\nabla\omega$ since

$$\langle L, \underline{L}\rangle = \langle l, \underline{L}\rangle = 2$$
$$\langle L, \tilde{V}\rangle = \langle l, V\omega\underline{L}\rangle - 2\langle\nabla\omega, V\rangle = 2V\omega - 2V\omega = 0$$

giving

$$\begin{aligned}
\zeta(\tilde{V}) &= \frac{1}{2}\langle D_{V+V\omega\underline{L}}\underline{L}, l - |\nabla\omega|^2\underline{L} - 2\nabla\omega\rangle \\
&= \frac{1}{2}\langle D_V\underline{L} + \kappa V\omega\underline{L}, l - |\nabla\omega|^2\underline{L} - 2\nabla\omega\rangle \\
&= t(V) - \frac{1}{4}|\nabla\omega|^2 V\langle\underline{L}, \underline{L}\rangle - \langle D_V\underline{L}, \nabla\omega\rangle + \kappa\langle V, \nabla\omega\rangle \\
&= t(V) - K(V, \nabla\omega) + \kappa\langle V, \nabla\omega\rangle.
\end{aligned}$$

For comparison between χ and Q we calculate

$$\chi(\tilde{V}, \tilde{W}) = \langle D_{V+V\omega\underline{L}}(l - |\nabla\omega|^2\underline{L} - 2\nabla\omega), W + W\omega\underline{L}\rangle$$

in three parts:

$$\begin{aligned}
\langle D_{V+V\omega\underline{L}}l, W+W\omega\underline{L}\rangle &= Q(V,W)+V\omega\langle D_{\underline{L}}l, W\rangle + W\omega\langle D_V l, \underline{L}\rangle \\
&\quad + V\omega W\omega\langle D_{\underline{L}}l, \underline{L}\rangle \\
&= Q(V,W) - V\omega\langle l, D_{\underline{L}}W\rangle - W\omega\langle l, D_V\underline{L}\rangle \\
&\quad - V\omega W\omega\langle l, D_{\underline{L}}\underline{L}\rangle \\
&= Q(V,W) - 2V\omega t(W) - 2W\omega t(V) - 2\kappa V\omega W\omega \\
-|\nabla\omega|^2\langle D_{V+V\omega\underline{L}}\underline{L}, W+W\omega\underline{L}\rangle &= -|\nabla\omega|^2\langle D_V\underline{L}, W+W\omega\underline{L}\rangle \\
&= -|\nabla\omega|^2K(V,W) - \frac{1}{2}|\nabla\omega|^2 W\omega V\langle\underline{L},\underline{L}\rangle \\
&= -|\nabla\omega|^2K(V,W) \\
-2\langle D_{V+V\omega\underline{L}}\nabla\omega, W+W\omega\underline{L}\rangle &= -2\langle D_V\nabla\omega, W\rangle - 2W\omega\langle D_V\nabla\omega,\underline{L}\rangle \\
&\quad - 2V\omega\langle D_{\underline{L}}\nabla\omega, W\rangle - 2V\omega W\omega\langle D_{\underline{L}}\nabla\omega,\underline{L}\rangle \\
&= -2H^\omega(V,W) + 2W\omega K(\nabla\omega, V) - 2V\omega\underline{L}W\omega \\
&\quad + 2V\omega K(\nabla\omega, W) + 2V\omega W\omega\langle\nabla\omega, D_{\underline{L}}\underline{L}\rangle \\
&= -2H^\omega(V,W) + 2W\omega K(\nabla\omega,V) + 2V\omega K(\nabla\omega,W)
\end{aligned}$$

the third to last line coming from

$$\langle D_{\underline{L}}\nabla\omega, W\rangle = \underline{L}\langle\nabla\omega, W\rangle - \langle\nabla\omega, D_{\underline{L}}W\rangle = \underline{L}W\omega - K(W,\nabla\omega)$$

Collecting all the terms the result follows. The final identity follows upon taking a trace. □

5.2 Flux Comparison

We are now ready to prove our first main result of this section. On Σ_ω we will denote the flux function (2.1) by $\not\rho$ and on Σ_s by ρ the following theorem provides comparison between the two

Theorem 5.2.1 (Flux Comparison Theorem). *At any $q \in \Sigma_\omega \cap \Sigma_s$ we have*

$$\not{\phi} = \rho + \not{\nabla} \cdot \Big(\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K} \not{\nabla}\omega\Big) + \frac{1}{2}\Big(|\hat{K}|^2 + G(\underline{L}, \underline{L})\Big)|\nabla\omega|^2$$

$$+ \nabla\omega \frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K} + G(\underline{L}, \nabla\omega) - 2\hat{K}(\vec{t} - \nabla \log \operatorname{tr} K, \nabla\omega)$$

Remark 5.2.1. *Revisiting Theorem 4.3.2 and the case that $\hat{\underline{\chi}} = G(\underline{L}, \cdot) = 0$, Theorem 5.2.1 provides an alternative proof that $\not{\phi}$ agrees with ρ point wise.*

Proof. When used, we assume $V, W, U \in E(\Sigma_{s_0})(\implies \tilde{V}, \tilde{W}, \tilde{U} \in E(\Sigma_\omega))$. We will need to know how to relate the covariant derivatives between the two surfaces so first a lemma

Lemma 5.2.1. $T_\omega\Big(\nabla_V W + V\omega\vec{K}(W) + W\omega\vec{K}(V) - K(V, W)\nabla\omega\Big) = \not{\nabla}_{\tilde{V}}\tilde{W}$

Proof. Since $\not{\nabla}_{\tilde{V}}\tilde{W}|_q = (S + S\omega\underline{L})|_q = T_\omega(S|_q)$ for some $S \in \Gamma(T\Sigma_{s(q)})$ it follows that $\langle \not{\nabla}_{\tilde{V}}\tilde{W}, U\rangle = \langle S, U\rangle$ for any $U \in E(\Sigma_{s_0})$. We find

$$\begin{aligned}
\langle \not{\nabla}_{\tilde{V}}\tilde{W}, U\rangle &= \langle D_{\tilde{V}}\tilde{W} + \frac{1}{2}\underline{\chi}(\tilde{V}, \tilde{W})L + \frac{1}{2}\chi(\tilde{V}, \tilde{W})\underline{L}, U\rangle \\
&= \langle D_{\tilde{V}}\tilde{W}, U\rangle + \frac{1}{2}K(V, W)\langle L, U\rangle \\
&= \tilde{V}\langle W, U\rangle - \langle \tilde{W}, D_{\tilde{V}}U\rangle + \frac{1}{2}K(V, W)\langle l - |\nabla\omega|^2\underline{L} - 2\nabla\omega, U\rangle \\
&= (V + V\omega\underline{L})\langle W, U\rangle - \langle W + W\omega\underline{L}, D_{V+V\omega\underline{L}}U\rangle - K(V, W)U\omega \\
&= V\langle W, U\rangle + 2V\omega K(W, U) - \Big(\langle W, \nabla_V U\rangle + V\omega K(W, U) - W\omega K(V, U)\Big) \\
&\quad - K(V, W)U\omega \\
&= \Big(V\langle W, U\rangle - \langle W, \nabla_V U\rangle\Big) + K(W, U)V\omega + K(V, U)W\omega - K(V, W)U\omega \\
&= \langle \nabla_V W + V\omega\vec{K}(W) + W\omega\vec{K}(V) - K(V, W)\nabla\omega, U\rangle
\end{aligned}$$

so $S = \nabla_V W + V\omega\vec{K}(W) + W\omega\vec{K}(V) - K(V, W)\nabla\omega$ since $E(\Sigma_{s_0})|_{\Sigma_{s(q)}} = \Gamma(T\Sigma_{s(q)})$.

□

Now we proceed with the proof of Theorem 5.2.1 in 3 parts:

Step 1 *Comparison between* $\not\nabla \cdot \zeta$ *and* $\nabla \cdot t$*:*

From Lemmas 5.1.1 and 5.2.1 we have

$$\begin{aligned}
(\not\nabla_{\tilde{V}}\zeta)(\tilde{W}) &= \tilde{V}(\zeta(\tilde{W})) - \zeta(\not\nabla_{\tilde{V}}\tilde{W}) \\
&= (V + V\omega\underline{L})\Big(t(W) - K(W, \nabla\omega) + \kappa\langle W, \nabla\omega\rangle\Big) \\
&\quad - t\Big(\nabla_V W + V\omega\vec{K}(W) + W\omega\vec{K}(V) - K(V,W)\nabla\omega\Big) \\
&\quad + K\Big(\nabla_V W + V\omega\vec{K}(W) + W\omega\vec{K}(V) - K(V,W)\nabla\omega, \nabla\omega\Big) \\
&\quad - \kappa\langle\nabla_V W + V\omega\vec{K}(W) + W\omega\vec{K}(V) - K(V,W)\nabla\omega, \nabla\omega\rangle.
\end{aligned}$$

Isolating the terms of the second line we get

$$\begin{aligned}
&(V + V\omega\underline{L})(t(W) - K(W, \nabla\omega) + \kappa W\omega) \\
&\quad = Vt(W) + V\omega\Big(G_{\underline{L}}(W) - \nabla\cdot\hat{K}(W) - \operatorname{tr} K t(W) + \frac{1}{2}W \operatorname{tr} K + W\kappa\Big) \\
&\quad - VK(W, \nabla\omega) - V\omega(\mathcal{L}_{\underline{L}}K)(W, \nabla\omega) - V\omega K(W, [\underline{L}, \nabla\omega]) \\
&\quad + V\kappa W\omega + \kappa VW\omega + V\omega\underline{L}\kappa W\omega
\end{aligned}$$

where (4.7) was used to give the first line. To continue we'll need an expression for $[\underline{L}, \nabla\omega]$ and use (4.2) to get it:

$$\begin{aligned}
2K(\nabla\omega, V) = (\mathcal{L}_{\underline{L}}\gamma_s)(\nabla\omega, V) &= \underline{L}\langle\nabla\omega, V\rangle - \langle[\underline{L}, \nabla\omega], V\rangle \\
&= \underline{L}V\omega - \langle[\underline{L}, \nabla\omega], V\rangle \\
&= -\langle[\underline{L}, \nabla\omega], V\rangle
\end{aligned}$$

since $[\underline{L}, \nabla\omega] \in \Gamma(T\Sigma_s)$ we conclude that $[\underline{L}, \nabla\omega] = -2\vec{K}(\nabla\omega)$. Substitution back into our calculation and using (4.3) in the form

$$\mathcal{L}_{\underline{L}}K(V,W) = -\underline{\alpha}(V,W) + \langle\vec{K}(V), \vec{K}(W)\rangle + \kappa K(V,W)$$

gives

$$(V + V\omega\underline{L})(t(W) - K(W,\nabla\omega) + \kappa W\omega) = Vt(W) - VK(W,\nabla\omega)$$

$$+ V\omega\Big(G_{\underline{L}}(W) - \nabla\cdot\hat{K}(W) - \operatorname{tr}Kt(W) + \frac{1}{2}W\operatorname{tr}K + \underline{\alpha}(W,\nabla\omega) + \langle\vec{K}(W),\vec{K}(\nabla\omega)\rangle\Big)$$

$$+ V\omega W\kappa - \kappa V\omega K(W,\nabla\omega) + V\kappa W\omega + \kappa VW\omega + \underline{L}\kappa V\omega W\omega.$$

Collecting terms we get

$$(\tilde{\nabla}_{\tilde{V}}\zeta)(\tilde{W}) = Vt(W) - t(\nabla_V W) + K(\nabla_V W,\nabla\omega) - VK(W,\nabla\omega)$$

$$+ V\omega\Big(G_{\underline{L}}(W) - \nabla\cdot\hat{K}(W) - \operatorname{tr}Kt(W) + \frac{1}{2}W\operatorname{tr}K + \underline{\alpha}(W,\nabla\omega) + \langle\vec{K}(W),\vec{K}(\nabla\omega)\rangle\Big)$$

$$- V\omega K(W,\vec{t}) - W\omega K(V,\vec{t}) + K(V,W)t(\nabla\omega)$$

$$+ V\omega\langle\vec{K}(W),\vec{K}(\nabla\omega)\rangle + W\omega\langle\vec{K}(V),\vec{K}(\nabla\omega)\rangle - K(V,W)K(\nabla\omega,\nabla\omega)$$

$$+ V\omega W\kappa - \kappa V\omega K(W,\nabla\omega) + V\kappa W\omega + \kappa VW\omega + \underline{L}\kappa V\omega W\omega$$

$$- \kappa\nabla_V W\omega - \kappa V\omega K(W,\nabla\omega) - \kappa W\omega K(V,\nabla\omega) + \kappa K(V,W)|\nabla\omega|^2.$$

So taking a trace over V and W

$$\tilde{\nabla}\cdot\zeta = \nabla\cdot t - \nabla\cdot(\vec{K}(\nabla\omega))$$

$$+ \Big(G_{\underline{L}}(\nabla\omega) - (\nabla\cdot\hat{K})(\nabla\omega) - \operatorname{tr}Kt(\nabla\omega) + \frac{1}{2}\nabla\omega\operatorname{tr}K + \underline{\alpha}(\nabla\omega,\nabla\omega) + |\vec{K}(\nabla\omega)|^2\Big)$$

$$- 2K(\nabla\omega,\vec{t}) + \operatorname{tr}Kt(\nabla\omega) + 2|\vec{K}(\nabla\omega)|^2 - \operatorname{tr}KK(\nabla\omega,\nabla\omega)$$

$$+ 2\nabla\omega\kappa - 3\kappa K(\nabla\omega,\nabla\omega) + \kappa\Delta\omega + \underline{L}\kappa|\nabla\omega|^2 + \kappa\operatorname{tr}K|\nabla\omega|^2$$

$$= \nabla\cdot t - \Big(\nabla\cdot(\vec{K}(\nabla\omega)) + (\nabla\cdot\hat{K})(\nabla\omega) - \frac{1}{2}\nabla\omega\operatorname{tr}K\Big) - 2\Big(K(\nabla\omega,\vec{t}) - \frac{1}{2}\operatorname{tr}Kt(\nabla\omega)\Big)$$

$$+ 3|\vec{K}(\nabla\omega)|^2 - \operatorname{tr}KK(\nabla\omega,\nabla\omega) + G_{\underline{L}}(\nabla\omega) - \operatorname{tr}Kt(\nabla\omega) + \underline{\alpha}(\nabla\omega,\nabla\omega)$$

$$+ 2\nabla\omega\kappa - 3\kappa\hat{K}(\nabla\omega,\nabla\omega) + \kappa\Delta\omega + \underline{L}\kappa|\nabla\omega|^2 - \frac{1}{2}\kappa\operatorname{tr}K|\nabla\omega|^2$$

$$= \nabla\cdot t - \Big(2(\nabla\cdot\hat{K})(\nabla\omega) + H^\omega\cdot K\Big) - 2\hat{K}(\nabla\omega,\vec{t}) + 3|\vec{K}(\nabla\omega)|^2 - \operatorname{tr}KK(\nabla\omega,\nabla\omega)$$

$$+ G_{\underline{L}}(\nabla\omega) - \operatorname{tr}Kt(\nabla\omega) + \underline{\alpha}(\nabla\omega,\nabla\omega)$$

$$+ 2\nabla\omega\kappa - 3\kappa\hat{K}(\nabla\omega,\nabla\omega) + \kappa\Delta\omega + \underline{L}\kappa|\nabla\omega|^2 - \frac{1}{2}\kappa\,\mathrm{tr}\,K|\nabla\omega|^2$$

$$= \nabla\cdot t - 2(\nabla\cdot\hat{K})(\nabla\omega) - H^{\omega}\cdot\hat{K} - \frac{1}{2}\,\mathrm{tr}\,K\Delta\omega - 2\hat{K}(\nabla\omega,\vec{t}) + \frac{3}{2}|\hat{K}|^2|\nabla\omega|^2$$

$$+ 2\,\mathrm{tr}\,K\hat{K}(\nabla\omega,\nabla\omega) + \frac{1}{4}(\mathrm{tr}\,K)^2|\nabla\omega|^2 + G_{\underline{L}}(\nabla\omega) - \mathrm{tr}\,Kt(\nabla\omega) + \underline{\alpha}(\nabla\omega,\nabla\omega)$$

$$+ 2\nabla\omega\kappa - 3\kappa\hat{K}(\nabla\omega,\nabla\omega) + \kappa\Delta\omega + \underline{L}\kappa|\nabla\omega|^2 - \frac{1}{2}\kappa\,\mathrm{tr}\,K|\nabla\omega|^2.$$

Step 2 *Comparison between* $\nabla\!\!\!/\cdot\zeta - \Delta\!\!\!/\log\mathrm{tr}\,\underline{\chi}$ *and* $\nabla\cdot t - \Delta\log\mathrm{tr}\,K$:

Since $\mathrm{tr}\,\underline{\chi} = \mathrm{tr}\,K|_{\Sigma_\omega}$ we start by comparing $\Delta\!\!\!/\log\mathrm{tr}\,K$ with $\Delta\log\mathrm{tr}\,K$

$$H^{\log\mathrm{tr}\,\underline{\chi}}(\tilde{V},\tilde{W}) = \langle\nabla\!\!\!/_{\tilde{V}}\nabla\!\!\!/\log\mathrm{tr}\,K,\tilde{W}\rangle = \tilde{V}\tilde{W}\log\mathrm{tr}\,K - \nabla\!\!\!/_{\tilde{V}}\tilde{W}\log\mathrm{tr}\,K$$

So isolating the first term we get

$$\tilde{V}\tilde{W}\log\mathrm{tr}\,K = (V + V\omega\underline{L})(W + W\omega\underline{L})\log\mathrm{tr}\,K$$

$$= VW\log\mathrm{tr}\,K + (VW\omega + V\omega W + W\omega V)\underline{L}\log\mathrm{tr}\,K + V\omega W\omega\underline{L}\underline{L}\log\mathrm{tr}\,K$$

and then the second

$$\nabla\!\!\!/_{\tilde{V}}\tilde{W}\log\mathrm{tr}\,K = (\nabla_V W + V\omega\vec{K}(W) + W\omega\vec{K}(V) - K(V,W)\nabla\omega)\log\mathrm{tr}\,K$$

$$+ (\nabla_V W + V\omega\vec{K}(W) + W\omega\vec{K}(V) - K(V,W)\nabla\omega)\omega\underline{L}\log\mathrm{tr}\,K$$

having used Lemma 5.2.1. Collecting terms

$$H^{\log\mathrm{tr}\,K}(\tilde{V},\tilde{W}) = VW\log\mathrm{tr}\,K - \nabla_V W\log\mathrm{tr}\,K$$

$$+ (VW\omega - \nabla_V W\omega)\underline{L}\log\mathrm{tr}\,K + V\omega W\omega\underline{L}\underline{L}\log\mathrm{tr}\,K$$

$$- \Big(V\omega K(W,\nabla\log\mathrm{tr}\,K) + W\omega K(V,\nabla\log\mathrm{tr}\,K) - K(V,W)\langle\nabla\omega,\nabla\log\mathrm{tr}\,K\rangle\Big)$$

$$+ \Big(K(V,W)|\nabla\omega|^2 - V\omega K(W,\nabla\omega) - W\omega K(V,\nabla\omega) + V\omega W + W\omega V\Big)\underline{L}\log\mathrm{tr}\,K.$$

So that a trace over V and W yields

$$\Delta\!\!\!/\log\mathrm{tr}\,K = \Delta\log\mathrm{tr}\,K + \Delta\omega\underline{L}\log\mathrm{tr}\,K + |\nabla\omega|^2\underline{L}\underline{L}\log\mathrm{tr}\,K - 2\hat{K}(\nabla\omega,\nabla\log\mathrm{tr}\,K)$$

$$- 2\hat{K}(\nabla\omega, \nabla\omega)\underline{L}\log\operatorname{tr}K + 2\nabla\omega\underline{L}\log\operatorname{tr}K.$$

We take the opportunity at this point of the calculation to bring to the attention of the reader that we have not yet used any distinguishing characteristics of the function $\log\operatorname{tr}K$ in comparison to an arbitrary $f \in \mathcal{F}(\Omega)$. In particular, we notice if $f \in \mathcal{F}(\Omega)$ satisfies $\underline{L}f = 0$, switching with $\log\operatorname{tr}K$ above, yields

$$\not\Delta f = \Delta f - 2\hat{K}(\nabla\omega, \nabla f).$$

As a result,

Lemma 5.2.2.

$$\not\Delta g = \Delta g + \not\nabla \cdot (\underline{L}g\not\nabla\omega) + \nabla\omega\underline{L}g - 2\hat{K}(\nabla\omega, \nabla g)$$

for any $g \in \mathcal{F}(\Omega)$.

Proof. We have

$$\begin{aligned}
\not\Delta g &= \Delta g + \Delta\omega\underline{L}g + |\nabla\omega|^2\underline{L}\underline{L}g - 2\hat{K}(\nabla\omega, \nabla g) - 2\hat{K}(\nabla\omega, \nabla\omega)\underline{L}g + 2\nabla\omega\underline{L}g \\
&= \Delta g + (\Delta\omega - 2\hat{K}(\nabla\omega, \nabla\omega))\underline{L}g + (\nabla\omega + |\nabla\omega|^2\underline{L})\underline{L}g + \nabla\omega\underline{L}g - 2\hat{K}(\nabla\omega, \nabla g) \\
&= \Delta g + \not\Delta\omega\underline{L}g + \not\nabla\omega\underline{L}g + \nabla\omega\underline{L}g - 2\hat{K}(\nabla\omega, \nabla g) \\
&= \Delta g + \not\nabla \cdot (\underline{L}g\not\nabla\omega) + \nabla\omega\underline{L}g - 2\hat{K}(\nabla\omega, \nabla g)
\end{aligned}$$

having used the fact that $\underline{L}\omega = 0$ and the comment immediately preceding the statement of Lemma 5.2.2 to get the third equality. □

Finishing up Step 2 we have

$$\begin{aligned}
\not\nabla \cdot \zeta - \not\Delta\log\operatorname{tr}\underline{\chi} &= \nabla \cdot t - \Delta\log\operatorname{tr}K \\
&\quad - 2(\nabla \cdot \hat{K})(\nabla\omega) - H^{\omega} \cdot \hat{K} - 2\hat{K}(\nabla\omega, \vec{t} - \nabla\log\operatorname{tr}K) \\
&\quad - \operatorname{tr}Kt(\nabla\omega) + \nabla\omega\operatorname{tr}K + \frac{3}{2}|\hat{K}|^2|\nabla\omega|^2 + G_{\underline{L}}(\nabla\omega) + \underline{\alpha}(\nabla\omega, \nabla\omega)
\end{aligned}$$

$$
\begin{aligned}
&- \Big(\frac{1}{2}\operatorname{tr} K \Delta\omega + \Delta\omega \underline{L} \log \operatorname{tr} K\Big) + \Big(\frac{1}{4}(\operatorname{tr} K)^2 - \underline{L}\underline{L} \log \operatorname{tr} K\Big)|\nabla\omega|^2 \\
&- \Big(\nabla\omega \operatorname{tr} K + 2\nabla\omega \underline{L} \log \operatorname{tr} K\Big) + 2\hat{K}(\nabla\omega, \nabla\omega)\Big(\operatorname{tr} K + \underline{L} \log \operatorname{tr} K\Big) \\
&+ 2\nabla\omega\kappa - 3\kappa\hat{K}(\nabla\omega, \nabla\omega) + \kappa\Delta\omega + \underline{L}\kappa|\nabla\omega|^2 - \frac{1}{2}\kappa \operatorname{tr} K|\nabla\omega|^2 \\
= {}& \nabla \cdot t - \Delta \log \operatorname{tr} K \\
&- 2(\nabla \cdot \hat{K})(\nabla\omega) - H^\omega \cdot \hat{K} - 2\hat{K}(\nabla\omega, \vec{t} - \nabla \log \operatorname{tr} K) \\
&- \operatorname{tr} K t(\nabla\omega) + \nabla\omega \operatorname{tr} K + |\hat{K}|^2|\nabla\omega|^2 + G_{\underline{L}}(\nabla\omega) + \underline{\hat{\alpha}}(\nabla\omega, \nabla\omega) \\
&+ \frac{1}{2}\Big(|\hat{K}|^2 + G(\underline{L}, \underline{L})\Big)|\nabla\omega|^2 + \Big(\Delta\omega - 2\hat{K}(\nabla\omega, \nabla\omega)\Big)\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K} \\
&+ \Big(- \frac{1}{2}(|\hat{K}|^2 + G(\underline{L}, \underline{L}) - \kappa \operatorname{tr} K) + \underline{L}\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K}\Big)|\nabla\omega|^2 \\
&+ 2\nabla\omega\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K} + \operatorname{tr} K\hat{K}(\nabla\omega, \nabla\omega) - \kappa\hat{K}(\nabla\omega, \nabla\omega) - \frac{1}{2}\kappa \operatorname{tr} K|\nabla\omega|^2 \\
= {}& \nabla \cdot t - \Delta \log \operatorname{tr} K \\
&- 2(\nabla \cdot \hat{K})(\nabla\omega) - H^\omega \cdot \hat{K} - 2\hat{K}(\nabla\omega, \vec{t} - \nabla \log \operatorname{tr} K) \\
&- \operatorname{tr} K t(\nabla\omega) + \nabla\omega \operatorname{tr} K + |\hat{K}|^2|\nabla\omega|^2 + G_{\underline{L}}(\nabla\omega) + \underline{\hat{\alpha}}(\nabla\omega, \nabla\omega) \\
&+ \not{\Delta}\omega\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K} + \underline{L}\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K}|\nabla\omega|^2 + 2\nabla\omega\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K} \\
&+ \operatorname{tr} K\hat{K}(\nabla\omega, \nabla\omega) - \kappa\hat{K}(\nabla\omega, \nabla\omega)
\end{aligned}
$$

having used (8) to get the last two lines in the second equality, Lemma 5.2.2 to get $\Delta\omega - 2\hat{K}(\nabla\omega, \nabla\omega) = \not{\Delta}\omega$ in the second equality followed by cancellation of the terms $\frac{1}{2}\Big(|\hat{K}|^2 + G(\underline{L}, \underline{L})\Big)|\nabla\omega|^2$ and $\frac{1}{2}\kappa \operatorname{tr} K|\nabla\omega|^2$.

Step 3 *Comparison between* $\not{\phi}$ *and* ρ:

Denoting the Gauss curvature on Σ_s by $\mathcal{C}$ and the mean curvature vector $\vec{h}$ we have from the Gauss equation (3.1)

$$\mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H}\rangle + \frac{1}{2}\hat{\underline{\chi}} \cdot \hat{\chi} = -\frac{1}{2}R - G(\underline{L}, L) - \frac{1}{4}\langle R_{\underline{L}L}\underline{L}, L\rangle$$

$$= -\frac{1}{2}R - G(\underline{L}, l - |\nabla\omega|^2\underline{L} - 2\nabla\omega) - \frac{1}{4}\langle R_{\underline{L}\, l-|\nabla\omega|^2\underline{L}-2\nabla\omega}\underline{L}, l - |\nabla\omega|^2\underline{L} - 2\nabla\omega\rangle$$

$$= \mathcal{C} - \frac{1}{4}\langle \vec{h}, \vec{h}\rangle + \frac{1}{2}\hat{K} \cdot \hat{Q} + |\nabla\omega|^2 G(\underline{L}, \underline{L}) + 2G(\underline{L}, \nabla\omega) - \langle R_{\underline{L}\,\nabla\omega} l, \underline{L}\rangle$$

$$- \langle R_{\underline{L}\,\nabla\omega}\underline{L}, \nabla\omega\rangle$$

$$= \mathcal{C} - \frac{1}{4}\langle \vec{h}, \vec{h}\rangle + \frac{1}{2}\hat{K} \cdot \hat{Q} + \frac{1}{2}|\nabla\omega|^2 G(\underline{L}, \underline{L}) + \Big(2G(\underline{L}, \nabla\omega) - \langle R_{\underline{L}\,\nabla\omega} l, \underline{L}\rangle\Big)$$

$$- \hat{\underline{\alpha}}(\nabla\omega, \nabla\omega)$$

from this we conclude

$$\Big(\mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H}\rangle + \not{\nabla} \cdot \zeta - \not{\Delta} \log \operatorname{tr} \underline{\chi}\Big) - \Big(\mathcal{C} - \frac{1}{4}\langle \vec{h}, \vec{h}\rangle + \nabla \cdot t - \Delta \log \operatorname{tr} K\Big)$$

$$= \frac{1}{2}\Big(\hat{K} \cdot \hat{Q} - \hat{\underline{\chi}} \cdot \hat{\chi}\Big)$$

$$+ \frac{1}{2}|\nabla\omega|^2 G(\underline{L}, \underline{L}) + \Big(2G(\underline{L}, \nabla\omega) - \langle R_{\underline{L}\,\nabla\omega} l, \underline{L}\rangle\Big) - \hat{\underline{\alpha}}(\nabla\omega, \nabla\omega)$$

$$- 2(\nabla \cdot \hat{K})(\nabla\omega) - H^{\omega} \cdot \hat{K} - 2\hat{K}(\nabla\omega, \vec{t} - \nabla \log \operatorname{tr} K)$$

$$- \operatorname{tr} K t(\nabla\omega) + \nabla\omega \operatorname{tr} K + |\hat{K}|^2|\nabla\omega|^2 + G_{\underline{L}}(\nabla\omega) + \hat{\underline{\alpha}}(\nabla\omega, \nabla\omega)$$

$$+ \not{\Delta}\omega \frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K} + \underline{L}\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K}|\nabla\omega|^2 + 2\nabla\omega \frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K}$$

$$+ \operatorname{tr} K \hat{K}(\nabla\omega, \nabla\omega) - \kappa\hat{K}(\nabla\omega, \nabla\omega)$$

Isolating the first two terms and using Lemma 5.1.1 we get

$$\hat{K} \cdot \hat{Q} - \hat{\underline{\chi}} \cdot \hat{\chi}$$

$$= \hat{K} \cdot \hat{Q} - \Big(\hat{K} \cdot \hat{Q} - |\nabla\omega|^2|\hat{K}|^2 - 4\hat{K}(\nabla\omega, \vec{t}) + 2|\hat{K}|^2|\nabla\omega|^2$$

$$+ 2 \operatorname{tr} K \hat{K}(\nabla\omega, \nabla\omega) - 2\hat{K} \cdot H^{\omega} - 2\kappa\hat{K}(\nabla\omega, \nabla\omega)\Big)$$

$$= -|\hat{K}|^2|\nabla\omega|^2 - 2 \operatorname{tr} K \hat{K}(\nabla\omega, \nabla\omega) + 2\hat{K} \cdot H^{\omega} + 4\hat{K}(\nabla\omega, \vec{t}) + 2\kappa\hat{K}(\nabla\omega, \nabla\omega)$$

and finally we have

$$\Big(\mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H}\rangle + \not\nabla \cdot \zeta - \not\Delta \log \operatorname{tr} \underline{\chi}\Big) - \Big(\mathcal{C} - \frac{1}{4}\langle \vec{h}, \vec{h}\rangle + \nabla \cdot t - \Delta \log \operatorname{tr} K\Big)$$

$$= \frac{1}{2}|\nabla\omega|^2\Big(|\hat{K}|^2 + G(\underline{L}, \underline{L})\Big) + G_{\underline{L}}(\nabla\omega) - 2\hat{K}(\nabla\omega, \vec{t} - \nabla \log \operatorname{tr} K)$$

$$+ \Big(2G_{\underline{L}}(\nabla\omega) - \langle R_{\underline{L}\,\nabla\omega} l, \underline{L}\rangle - 2(\nabla \cdot \hat{K})(\nabla\omega) + 2\hat{K}(\nabla\omega, \vec{t}) - \operatorname{tr} K t(\nabla\omega) + \nabla\omega \operatorname{tr} K\Big)$$

$$+ \not\Delta\omega \frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K} + \underline{L}\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K}|\nabla\omega|^2 + 2\nabla\omega \frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K}.$$

Amazingly the third line vanishes by the Codazzi equation (3.2) as well as all terms with a factor κ giving

$$\not\rho - \rho = \frac{1}{2}\Big(|\hat{K}|^2 + G(\underline{L}, \underline{L})\Big)|\nabla\omega|^2 + G_{\underline{L}}(\nabla\omega) - 2\hat{K}(\nabla\omega, \vec{t} - \nabla \log \operatorname{tr} K)$$

$$+ \not\Delta\omega \frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K} + \underline{L}\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K}|\nabla\omega|^2 + 2\nabla\omega \frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K}$$

and the result then follows from the fact that $\not\nabla\omega = \nabla\omega + |\nabla\omega|^2\underline{L}$ as well as

$$\not\nabla \cdot \Big(\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K}\not\nabla\omega\Big) = \not\Delta\omega \frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K} + \not\nabla\omega \frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K}.$$

□

5.3 Asymptotic flatness

In this section we wish to study the limiting behaviour of our mass functional in the setting of asymptotic flatness constructed by Mars and Soria [18]. Beyond the assumption that we have a cross-section Σ_{s_0} of Ω we also assume for some (hence any) choice of past-directed geodesic null generator $\underline{L}$ (i.e. $D_{\underline{L}}\underline{L} = 0$) that $S_+ = \infty$. So all geodesics $\gamma_q^{\underline{L}}$ are 'past complete' with domain $(s_-(q), \infty)$. We now take $s_0 = 0$ ignoring all points p satisfying $s(p) \leqslant S_-$ and conclude that $\Omega \cong \mathbb{S}^2 \times (S_-, \infty)$. Although the value of S_- will depend on our choice of geodesic generator $\underline{L}$ our interest lies only on the past of Σ_0 (i.e. $\mathbb{S}^2 \times (0, \infty)$) so we ignore this subtlety. A null hypersurface Ω with all the above properties is called *extending to past null infinity.*

In order to impose decay conditions of various *transversal tensors* (i.e. tensors satisfying $T(\underline{L}, \cdots) = \cdots = T(\cdots, \underline{L}) = 0$) we choose a local basis on Σ_0 and extend it to a basis field $\{X_i\} \subset E(\Sigma_0)$. Given a transversal k-tensor $T(s)$ we say,

- $T = O(1)$ iff $T_{i_1 \ldots i_k} := T(X_{i_1}, ..., X_{i_k})$ is uniformly bounded and $T = O_n(s^{-m})$ iff
$$s^{m+j}(\mathcal{L}_{\underline{L}})^j T(s) = O(1) \quad (0 \leqslant j \leqslant n)$$
- $T = o(s^{-m})$ iff $\lim_{s\to\infty} s^m T(s)_{i_1\ldots i_k} = 0$ and $T = o_n(s^{-m})$ iff
$$s^{m+j}(\mathcal{L}_{\underline{L}})^j T(s) = o(1) \quad (0 \leqslant j \leqslant n)$$
- $T = o_n^X(s^{-m})$ iff
$$s^m \mathcal{L}_{X_{i_1}} \cdots \mathcal{L}_{X_{i_j}} T(s) = o(1) \quad (0 \leqslant j \leqslant n).$$

Now we're ready to define asymptotic flatness for Ω as given by the authors of [18]:

Definition 5.3.1. *We say Ω is past asymptotically flat if it extends to past null infinity and there exists a choice of cross-section Σ_0 and null geodesic generator $\underline{L}$ with corresponding level set function s satisfying the following:*

1. *There exists two symmetric 2-covariant transversal and $\underline{L}$ Lie constant tensor fields $\mathring{\gamma}$ and γ_1 such that*
$$\tilde{\gamma} := \gamma - s^2\mathring{\gamma} - s\gamma_1 = o_1(s) \cap o_2^X(s)$$

2. There exists a tansversal and $\underline{L}$ Lie constant one-form t_1 such that

$$\tilde{t} := t - \frac{t_1}{s} = o_1(s^{-1})$$

3. There exist $\underline{L}$ Lie constant functions θ_0 and θ such that

$$\tilde{\theta} := \operatorname{tr} Q - \frac{\theta_0}{s} - \frac{\theta}{s^2} = o(s^{-2})$$

4. The scalar $\langle R_{X_{i_1} X_{i_2}} X_{i_3}, X_{i_4} \rangle$ along Ω is such that $\lim_{s\to\infty} \frac{1}{s^2} \langle R_{X_{i_1} X_{i_2}} X_{i_3}, X_{i_4} \rangle$ exists while its double trace satisfies $-\frac{1}{2}R - G(\underline{L}, l) - \frac{1}{4}\langle R_{\underline{L}l}\underline{L}, l\rangle = o(s^{-2})$.

We will have the need to supplement the notion of asymptotic flatness of Ω with a stronger version of the *energy flux decay condition* ($G_{\underline{L}} = o(s^{-2})$, $\mathcal{L}_{\underline{L}}\tilde{\gamma} = o_1^X(1)$ as given in [18]) with the following:

Definition 5.3.2. *Suppose Ω is past asymptotically flat. We say Ω has strong flux decay if*

$$G_{\underline{L}} = o(s^{-2}),\ \tilde{t} = o_1^X(s^{-1}) \text{ and } \mathcal{L}^j_{\underline{L}}\tilde{\gamma} = o^X_{3-j}(s^{1-j}) \text{ for } 1 \leqslant j \leqslant 3$$

and strong decay if the condition on $G_{\underline{L}}$ is dropped.

We will also need some results from [18] (Proposition 3, Lemma 2, Section 4) resulting directly from the asymptotically flat restriction on Ω. One particularly valuable consequence is the ability to choose our geodesic generator $\underline{L}$ to give any conformal change on the 'metric at null infinity', which turns out to be given by the 2-tensor, $\mathring{\gamma}$. By the Uniformization Theorem we conclude that this covers all possible metrics on a Riemannian 2-sphere. We will denote the covariant derivative coming from $\mathring{\gamma}$ by $\mathring{\nabla}$.

Proposition 5.3.1. *Suppose Ω is past asymptotically flat with a choice of affinely parametrized null generator $\underline{L}$ and corresponding level set function s. Letting $\gamma(s)^{ij}$ denote the inverse of $\gamma(s)_{ij}$,*

$$\gamma(s)^{ij} = \frac{1}{s^2}\mathring{\gamma}^{ij} - \frac{1}{s^3}\mathring{\gamma_1}^{ij} + o(s^{-3}) \tag{5.1}$$

$$K_{ij} = s\mathring{\gamma}_{ij} + \frac{1}{2}\gamma_{1ij} + o(1) \tag{5.2}$$

$$\mathcal{K}_{\gamma(s)} = \frac{\mathring{\mathcal{K}}}{s^2} + o(s^{-2}) \tag{5.3}$$

$$\operatorname{tr} Q = \frac{2\mathring{\mathcal{K}}}{s} + \frac{\theta}{s^2} + o(s^{-2}) \tag{5.4}$$

$$\operatorname{tr} K = \frac{2}{s} + \frac{\theta}{s^2} + o(s^{-2}) \tag{5.5}$$

where $\mathring{\gamma}^{ij}$ is the inverse of $\mathring{\gamma}_{ij}$, tensors with ring highlight the fact that indices have been raised with $\mathring{\gamma}$ and $\underline{\theta} = -\frac{1}{2}\mathring{\operatorname{tr}}\gamma_1$.
It follows in case $\mathcal{L}_{\underline{L}}\tilde{\gamma} = o_1^X(1)$ that

$$t_1 = \frac{1}{2}\mathring{\nabla}\cdot\gamma_1 + \not{d}\underline{\theta} \iff G_{\underline{L}} = o(s^{-2})$$

Proof. We refer the reader to [18] (Proposition 3) for proof. □

As promised in Remark 2.1.1 we are now able to prove the following well known result:

Lemma 5.3.1. *Suppose Ω extends to past null infinity with null geodesic generator $\underline{L}$. Then any cross-section $\Sigma \hookrightarrow \Omega$ satisfies $\operatorname{tr} K \geqslant 0$. If Ω is past asymptotically flat then Σ is expanding along $\underline{L}$.*

Proof. For $\omega \in \mathcal{F}(\Omega)$ constructed by Lie dragging $s|_\Sigma$ along $\underline{L}$ we have $\Sigma = \Sigma_1$ for the geodesic foliation $\{\Sigma_\lambda\}$ given by $s = \omega\lambda$. So it suffices to prove the result along an arbitrary geodesic foliation for Ω. From (8) we have, whenever $\operatorname{tr} K(s_0) < 0$ for some s_0, that

$$\underline{L}\Big(\frac{1}{\operatorname{tr} K}\Big) = \frac{1}{2} + |\hat{\chi}^-|^2 + G(L^-, L^-) \geqslant \frac{1}{2}$$

wherever it be defined as well as

$$\frac{1}{\operatorname{tr} K}(s) \geqslant \frac{1}{\operatorname{tr} K}(s_0) + \frac{s - s_0}{2}$$

for any such $s \geqslant s_0$. So we can find an $s_1 > s_0$ such that $\operatorname{tr} K(s) \xrightarrow{s \to s_1^-} -\infty$. Since this contradicts smoothness we must have that $\operatorname{tr} K \geqslant 0$ on all of Ω. If Ω is past asymptotically flat it follows from Proposition 5.3.1 that $\operatorname{tr} K(s) > 0$ for sufficiently large s. Since (4.4) gives

$$\underline{L}(\operatorname{tr} K) = -\frac{1}{2}(\operatorname{tr} K)^2 - |\hat{K}|^2 - G(\underline{L}, \underline{L}) \leqslant 0$$

we have $\operatorname{tr} K(s_0) \geqslant \operatorname{tr} K(s_1)$ for all $s_0 \leqslant s_1$. So we must have that $\operatorname{tr} K > 0$ on all of Ω. □

Lemma 5.3.2. *On each Σ_s the difference tensor*

$$\mathcal{D}(V, W) := \nabla_V W - \mathring{\nabla}_V W$$

admits the decomposition

$$\mathcal{D}^k_{ij} = \frac{1}{2}(\mathring{\nabla}_i \mathring{\gamma}_1{}^k{}_j + \mathring{\nabla}_j \gamma_1{}^k{}_i - \mathring{\nabla}^k \gamma_{1ij})\frac{1}{s} + O(s^{-2}).$$

Moreover, if $f \in \mathcal{F}(\Omega)$ is Lie constant along $\underline{L}$ then

$$\Delta f = \frac{1}{s^2}\mathring{\Delta} f + (-\mathring{\gamma}_1^{ij}\mathring{\nabla}_i\mathring{\nabla}_j f - (\mathring{\nabla}_i \mathring{\gamma}_1^{ij}) f_{,j} + (\mathring{\nabla}^i \underline{\theta}) f_{,i})\frac{1}{s^3} + o(s^{-3}).$$

Proof. The result follows from the well known fact (see, for example, [29]) that

$$\langle \mathcal{D}(V, W), U \rangle = \frac{1}{2}(\mathring{\nabla}_V \gamma(W, U) + \mathring{\nabla}_W \gamma(V, U) - \mathring{\nabla}_U \gamma(V, W))$$

$$= \frac{s}{2}(\mathring{\nabla}_V\gamma_1(W,U) + \mathring{\nabla}_W\gamma_1(V,U) - \mathring{\nabla}_U\gamma_1(V,W))$$

$$+ \frac{1}{2}(\mathring{\nabla}_V\tilde{\gamma}(W,U) + \mathring{\nabla}_W\tilde{\gamma}(V,U) - \mathring{\nabla}_U\tilde{\gamma}(V,W)).$$

The second is a simple consequence of the first, we refer the reader to [18] (Lemma 2) for proof. □

In the next Proposition we show that the decomposition of the metric given in Definition 5.3.1 part 1 allows us to find $\mathcal{K}_{\gamma(s)}$ up to $O(s^{-4})$:

Proposition 5.3.2. *For a decomposition of the metric $\gamma(s) = s^2\mathring{\gamma} + s\gamma_1 + \tilde{\gamma}$ for some fixed s we have:*

$$\mathcal{K}_{\gamma(s)} = \frac{\mathring{\mathcal{K}}}{s^2} + \frac{1}{s^3}(\mathring{\mathcal{K}}\underline{\theta} + \frac{1}{2}\mathring{\nabla}\cdot\mathring{\nabla}\cdot\gamma_1 + \mathring{\Delta}\underline{\theta}) + O(s^{-4}) \tag{5.6}$$

Proof. First we take the opportunity to show that $V, W \in E(\Sigma_0)$ gives $\mathring{\nabla}_V W \in E(\Sigma_0)$. Starting with the Koszul formula

$$2\mathring{\gamma}(\mathring{\nabla}_V W, U) = V\mathring{\gamma}(W,U) + W\mathring{\gamma}(U,V) - U\mathring{\gamma}(V,W)$$
$$- \mathring{\gamma}(V,[W,U]) + \mathring{\gamma}(W,[U,V]) + \mathring{\gamma}(U,[V,W])$$

and the fact that $\mathring{\gamma}$ is Lie constant along $\underline{L}$ we conclude that $\underline{L}\mathring{\gamma}(\mathring{\nabla}_V W, U) = \mathring{\gamma}([\underline{L}, \mathring{\nabla}_V W], U)$ on the left, applying $\underline{L}$ on the right we find everything vanishes since $V, W \in E(\Sigma_0) \implies [V,W] \in E(\Sigma_0)$. Therefore $\mathring{\gamma}([\underline{L}, \mathring{\nabla}_V W], U) = 0$. Since $[\underline{L}, \mathring{\nabla}_V W] \in \Gamma(T\Sigma_s)$ and $\mathring{\gamma}$ is positive definite it follows that $[\underline{L}, \mathring{\nabla}_V W] = 0$ and therefore $\mathring{\nabla}_V W \in E(\Sigma_0)$. To show the decomposition of $\mathcal{K}_{\gamma(s)}$ we start by finding the decomposition of the Riemann curvature tensor on Σ_s:

$$\langle R^s_{X_iX_j}X_k, X_m\rangle = \langle\nabla_{[X_i,X_j]}X_k, X_m\rangle - X_i\langle\nabla_{X_j}X_k, X_m\rangle + \langle\nabla_{X_j}X_k, \nabla_{X_i}X_m\rangle$$
$$+ X_j\langle\nabla_{X_i}X_k, X_m\rangle - \langle\nabla_{X_i}X_k, \nabla_{X_j}X_m\rangle$$
$$= \langle\mathring{\nabla}_{[X_i,X_j]}X_k, X_m\rangle - X_i\langle\mathring{\nabla}_{X_j}X_k, X_m\rangle + \langle\mathring{\nabla}_{X_j}X_k, \mathring{\nabla}_{X_i}X_m\rangle$$

$$
\begin{aligned}
&+ X_j\langle \mathring{\nabla}_{X_i}X_k, X_m\rangle - \langle \mathring{\nabla}_{X_i}X_k, \mathring{\nabla}_{X_j}X_m\rangle \\
&+ \langle \mathcal{D}([X_i,X_j],X_k),X_m\rangle - X_i\langle \mathcal{D}(X_j,X_k),X_m\rangle + \langle \mathcal{D}(X_j,X_k),\mathring{\nabla}_{X_i}X_m\rangle \\
&+ \langle \mathring{\nabla}_{X_j}X_k, \mathcal{D}(X_i,X_m)\rangle + X_j\langle \mathcal{D}(X_i,X_k),X_m\rangle - \langle \mathcal{D}(X_i,X_k),\mathring{\nabla}_{X_j}X_m\rangle \\
&- \langle \mathring{\nabla}_{X_i}X_k, \mathcal{D}(X_j,X_m)\rangle \\
&+ \langle \mathcal{D}(X_j,X_k),\mathcal{D}(X_i,X_m)\rangle - \langle \mathcal{D}(X_i,X_k),\mathcal{D}(X_j,X_m)\rangle.
\end{aligned}
$$

Using the decomposition $\gamma_s = s^2\mathring{\gamma} + O(s)$ we recognize the leading order term, combining lines 3 and 4, is $s^2\mathring{\gamma}(\mathring{R}_{X_iX_j}X_k, X_m)$. In order to find the next to leading order term the fact that $\langle R^s_{X_iX_j}X_k, X_m\rangle - s^2\mathring{\gamma}(\mathring{R}_{X_iX_j}X_k, X_m)$ defines a 4-tensor on each Σ_s allows us to search independently of our choice of basis $\{X_1, X_2\}$. In particular we may assume that $\mathring{\nabla}_{X_i}X_j = 0$ at $q \in \Sigma_s$ (hence on all of γ_q^L, since $\mathring{\nabla}_{X_i}X_j \in E(\Sigma_0)$). So assuming restriction to the generator through q and using Lemma 5.3.2 we have

$$
\begin{aligned}
\langle R^s_{X_iX_j}&X_k,X_m\rangle - s^2\mathring{\gamma}(\mathring{R}_{X_iX_j}X_k, X_m) \\
&= -sX_i\gamma_1(\mathring{\nabla}_{X_j}X_k, X_m) + sX_j\gamma_1(\mathring{\nabla}_{X_i}X_k, X_m) \\
&\quad - \frac{s}{2}X_i(\mathring{\nabla}_{X_j}\gamma_1(X_k,X_m) + \mathring{\nabla}_{X_k}\gamma_1(X_j,X_m) - \mathring{\nabla}_{X_m}\gamma_1(X_j,X_k)) \\
&\quad + \frac{s}{2}X_j(\mathring{\nabla}_{X_i}\gamma_1(X_k,X_m) + \mathring{\nabla}_{X_k}\gamma_1(X_i,X_m) - \mathring{\nabla}_{X_m}\gamma_1(X_i,X_k)) \\
&\quad + O(1) \\
&= -sX_i\gamma_1(\mathring{\nabla}_{X_j}X_k, X_m) + sX_j\gamma_1(\mathring{\nabla}_{X_i}X_k, X_m) \\
&\quad \frac{s}{2}\Big(\mathring{\nabla}_{X_j}\mathring{\nabla}_{X_i}\gamma_1(X_k,X_m) + \mathring{\nabla}_{X_j}\mathring{\nabla}_{X_k}\gamma_1(X_i,X_m) - \mathring{\nabla}_{X_j}\mathring{\nabla}_{X_m}\gamma_1(X_i,X_k) \\
&\quad - \mathring{\nabla}_{X_i}\mathring{\nabla}_{X_j}\gamma_1(X_k,X_m) - \mathring{\nabla}_{X_i}\mathring{\nabla}_{X_k}\gamma_1(X_j,X_m) + \mathring{\nabla}_{X_i}\mathring{\nabla}_{X_m}\gamma_1(X_j,X_k)\Big) \\
&\quad + O(1).
\end{aligned}
$$

It remains to simplify the two terms of the first line in the second equality. Since

$$
X_i\gamma_1(\mathring{\nabla}_{X_j}X_k, X_m) = \mathring{\nabla}_{X_i}\gamma_1(\mathring{\nabla}_{X_j}X_k, X_m) + \gamma_1(\mathring{\nabla}_{X_i}\mathring{\nabla}_{X_j}X_k, X_m)
$$

we conclude that

$$-X_i\gamma_1(\mathring{\nabla}_{X_j}X_k, X_m) + X_j\gamma_1(\mathring{\nabla}_{X_i}X_k, X_m) = \gamma_1(\mathring{R}_{X_iX_j}X_k, X_m).$$

Moreover, it is easily shown using our choice of basis extension that

$$\frac{1}{2}\mathring{\nabla}_{X_j}\mathring{\nabla}_{X_i}\gamma_1(X_k, X_m) - \frac{1}{2}\mathring{\nabla}_{X_i}\mathring{\nabla}_{X_j}\gamma_1(X_k, X_m) + \gamma_1(\mathring{R}_{X_iX_j}X_k, X_m)$$
$$= \frac{1}{2}(\gamma_1(\mathring{R}_{X_iX_j}X_k, X_m) - \gamma_1(\mathring{R}_{X_iX_j}X_m, X_k)).$$

So we finally have from the fact that Σ_s is of dimension 2 that

$$\begin{aligned}\langle R^s_{X_iX_j}X_k, X_m\rangle = s^2\mathring{\mathcal{K}}(\mathring{\gamma}_{ik}\mathring{\gamma}_{jm} - \mathring{\gamma}_{im}\mathring{\gamma}_{jk}) \\ + \frac{s}{2}\mathring{\mathcal{K}}(\mathring{\gamma}_{ik}\gamma_{1jm} - \mathring{\gamma}_{im}\gamma_{1jk} + \mathring{\gamma}_{jm}\gamma_{1ik} - \mathring{\gamma}_{jk}\gamma_{1im}) \\ + \frac{s}{2}(\mathring{\nabla}_j\mathring{\nabla}_k\gamma_{1im} - \mathring{\nabla}_j\mathring{\nabla}_m\gamma_{1ik} - \mathring{\nabla}_i\mathring{\nabla}_k\gamma_{1jm} + \mathring{\nabla}_i\mathring{\nabla}_m\gamma_{1jk}) + O(1).\end{aligned}$$

Using (12) to take a trace over i, k:

$$\begin{aligned}(Ric^s)_{jm} &= \mathring{\mathcal{K}}\mathring{\gamma}_{jm} - \frac{1}{s}\mathring{\mathcal{K}}\underline{\theta}\mathring{\gamma}_{jm} \\ &+ \frac{1}{2s}(\mathring{\nabla}_j(\mathring{\nabla}\cdot\gamma_1)_m + 2\mathring{\nabla}_j\mathring{\nabla}_m\underline{\theta} - (\mathring{\nabla}^2\gamma_1)_{jm} + (\mathring{\nabla}\cdot(\mathring{\nabla}\gamma_1))_{mj}) + \frac{\mathring{\mathcal{K}}}{s}(2\underline{\theta}\mathring{\gamma}_{jm} + \gamma_{1jm}) \\ &+ O(s^{-2}) \\ &= \mathring{\mathcal{K}}\mathring{\gamma}_{jm} \\ &+ \frac{1}{s}\Big(\mathring{\mathcal{K}}\underline{\theta}\mathring{\gamma}_{jm} + \mathring{\mathcal{K}}\gamma_{1jm} + \frac{1}{2}\mathring{\nabla}_j(\mathring{\nabla}\cdot\gamma_1)_m + \frac{1}{2}(\mathring{\nabla}\cdot(\mathring{\nabla}\gamma_1))_{mj} + \mathring{\nabla}_j\mathring{\nabla}_m\underline{\theta} - \frac{1}{2}(\mathring{\nabla}^2\gamma_1)_{jm}\Big) \\ &+ O(s^{-4})\end{aligned}$$

and then over j, m:

$$2\mathcal{K}_{\gamma(s)} = \frac{2}{s^2}\mathring{\mathcal{K}} + \frac{1}{s^3}\Big(2\mathring{\mathcal{K}}\underline{\theta} - 2\mathring{\mathcal{K}}\underline{\theta} + \mathring{\nabla}\cdot\mathring{\nabla}\cdot\gamma_1 + 2\mathring{\Delta}\underline{\theta}\Big) + \frac{2}{s^3}\mathring{\mathcal{K}}\underline{\theta} + O(s^{-4})$$

giving the result. □

Remark 5.3.1. *Interestingly, in the case that Ω is asymptotically flat satisfying the energy flux decay condition we conclude that*

$$\mathcal{K}_{\gamma(s)} = \frac{\mathring{\mathcal{K}}}{s^2} + \frac{1}{s^3}(\mathring{\mathcal{K}}\underline{\theta} + \mathring{\nabla} \cdot t_1) + O(s^{-4})$$

according to Proposition 5.3.2.

Definition 5.3.3. *For Ω past asymptotically flat with background geodesic foliation $\{\Sigma_s\}$ we say a foliation $\{\Sigma_{s_\star}\}$ is* <u>*asymptotically geodesic*</u> *provided*

$$s = \phi s_\star + \xi$$

with <u>*scale factor*</u> *$\phi > 0$ a Lie constant function along $\underline{L}$ and $\underline{L}^i \xi = o^X_{2-i}(s^{1-i})$ for $0 \leqslant i \leqslant 2$. In addition (similarly to [18]), we will say $\{\Sigma_{s_\star}\}$* <u>*approaches large spheres*</u> *provided the class of geodesic foliations measuring $\phi = 1$ also induce $\mathring{\gamma}$ to be the round metric on $\mathbb{S}^2$.*

Remark 5.3.2. *Given a basis extension $\{X_i\} \subset E(\Sigma_0)$ (on $\{\Sigma_s\}$) and a foliation $\{\Sigma_{s_\star}\}$ as in Definition 5.3.3, Lie dragging $s|_{\Sigma_{s_\star}}$ along $\underline{L}$ to give $\omega \in \mathcal{F}(\Omega)$ we see at $q \in \Sigma_{s_\star}$:*

$$\omega_i = \phi_i s_\star + \xi_s \omega_i + \xi_i$$

$$\omega_{ij} = \phi_{ij} s_\star + \xi_{ss}\omega_i\omega_j + \xi_{sj}\omega_i + \xi_s\omega_{ij} + \xi_{ij}$$

where $\omega_i := X_i\omega$, $\omega_{ij} := X_jX_i\omega$, $\xi_s := \underline{L}\xi$, $\xi_{ss} := \underline{L}\underline{L}\xi$, $\xi_i := X_i(\xi|_{\Sigma_{s(q)}})$, $\xi_{si} = X_i(\xi_s|_{\Sigma_s})$ and $\xi_{ij} = X_jX_i(\xi|_{\Sigma_s})$. The decay on ξ therefore gives us that:

$$\omega_i = \frac{\phi_i s_\star + \xi_i}{1 - \xi_s} = \phi_i s_\star + o(s_\star)$$

$$\begin{aligned}\omega_{ij} &= \frac{1}{1-\xi_s}\Big(\phi_{ij}s_\star + \xi_{ss}\Big(\frac{\xi_i + \phi_i s_\star}{1-\xi_s}\Big)\Big(\frac{\xi_j + \phi_j s_\star}{1-\xi_s}\Big) + \xi_{sj}\Big(\frac{\xi_i + \phi_i s_\star}{1-\xi_s}\Big) + \xi_{ij}\Big)\\ &= \phi_{ij}s_\star + o(s_\star).\end{aligned}$$

From (12) and Lemma 5.3.2 we conclude that

$$d\omega|_{\Sigma_{s_\star}} = (s_\star\phi)^2(\frac{-1}{s_\star}d\phi^{-1}|_{\Sigma_{s_\star}} + o(s_\star^{-1}))$$

$$\Delta\omega|_{\Sigma_{s_\star}} = \frac{1}{\phi^2 s_\star}\mathring{\Delta}\phi + o(s_\star^{-1}).$$

5.3.1 Asymptotic Mass and Energy

Using Theorem 5.2.1 we prove a slightly weakened version of the beautiful result found by the authors of [18] (Theorem 1):

Proposition 5.3.3. *Suppose Ω is past asymptotically flat and $\{\Sigma_{s_\star}\}$ is an asymptotically geodesic foliation with scale factor $\phi > 0$. Assuming $\mathcal{L}_{\underline{L}}\tilde{\gamma} = o_1^X(1)$ we have*

$$\lim_{s_\star\to\infty} E_H(\Sigma_{s_\star}) = \frac{1}{16\pi}\sqrt{\frac{\int \phi^2 d\mathring{A}}{4\pi}}\int \frac{1}{\phi}\Big(\mathring{\mathcal{K}}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla}\cdot t_1\Big)d\mathring{A}$$

with $\mathring{\gamma}$, $\mathring{\mathcal{K}}$, $\underline{\theta}$, θ and t_1 associated with the background geodesic foliation.

Proof. Given any fixed $s_\star$ we define $\omega \in \mathcal{F}(\Omega)$ by Lie dragging along $\underline{L}$:

$$s|_{\Sigma_{s_\star}} = (\phi s_\star + \xi)|_{\Sigma_{s_\star}}$$

as before. From the decomposition $\gamma_s = s^2\mathring{\gamma} + s\gamma_1 + \tilde{\gamma}$ and the standard identity for any invertible matrix M:

$$\det(M + sB) = \det M(1 + s\,\mathrm{tr}(M^{-1}B) + O(s^2))$$

we have

$$\sqrt{\det(\gamma_s)} = s^2\sqrt{\det(\mathring{\gamma})}(1 - \frac{1}{s}\underline{\theta} + o(s^{-1})).$$

From the first identity of Lemma 5.1.1 we therefore conclude that

$$dA_{s_\star} = dA_s|_{\Sigma_{s_\star}} = s_\star^2\phi^2 f d\mathring{A}$$

where $f = 1 + o(s_\star^0)$.

In Theorem 5.2.1, denoting the sum of all but the first two terms by $\eta_\omega(\nabla\omega)$ we see

$$\begin{aligned}\eta_\omega(\nabla\omega) &= \frac{1}{2}(|\hat{K}|^2 + G(\underline{L},\underline{L}))|\nabla\omega|^2 + G(\underline{L},\nabla\omega) + \nabla\omega\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr} K} \\ &\quad - 2\hat{K}(\vec{t} - \nabla\log\operatorname{tr} K, \nabla\omega) \\ &= \frac{1}{2}(|\hat{K}|^2 + G(\underline{L},\underline{L}))|\nabla\omega|^2 + G(\underline{L},\nabla\omega) - \nabla\omega(\underline{L}\log\operatorname{tr} K + \frac{1}{2}\operatorname{tr} K) \\ &\quad - 2\hat{K}(\vec{t} - \nabla\log\operatorname{tr} K, \nabla\omega)\end{aligned}$$

giving from Propositions 5.3.1 and 5.3.2

$$\begin{aligned}4\pi\frac{E_H(\Sigma_{s_\star})}{\sqrt{\frac{|\Sigma_{s_\star}|}{16\pi}}} &= \int \not\rho\, dA_\omega = \int \rho + \not\nabla\cdot\Big(\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr} K}\not\nabla\omega\Big) + \eta_\omega(\nabla\omega)dA_{s_\star} \\ &= \int \rho + \eta_\omega(\nabla\omega)dA_{s_\star} \\ &= \int \Big(\frac{\mathring{\mathcal{K}}}{\omega^2} + \frac{1}{\omega^3}\Big(\mathring{\mathcal{K}}\underline{\theta} + \frac{1}{2}\mathring{\nabla}\cdot\mathring{\nabla}\cdot\gamma_1 + \mathring{\Delta}\underline{\theta}\Big) - \frac{1}{4}(\frac{2}{\omega} + \frac{\underline{\theta}}{\omega^2})(\frac{2\mathring{\mathcal{K}}}{\omega} + \frac{\theta}{\omega^2}) \\ &\quad + \nabla\cdot t + G_{\underline{L}}(\nabla\omega) \\ &\quad + \frac{1}{2}(|\hat{K}|^2 + G(\underline{L},\underline{L}))|\nabla\omega|^2 \\ &\quad - \Delta\log\operatorname{tr} K - \nabla\omega(\underline{L}\log\operatorname{tr} K + \frac{1}{2}\operatorname{tr} K) - 2\hat{K}(\vec{t} - \nabla\log\operatorname{tr} K, \nabla\omega)\Big)dA_{s_\star} \\ &\quad + o(s_\star^{-1}) \\ &= \int \Big(\frac{1}{\omega^3}\Big(\frac{1}{2}\mathring{\mathcal{K}}\underline{\theta} - \frac{1}{2}\theta + \frac{1}{2}\mathring{\nabla}\cdot\mathring{\nabla}\cdot\gamma_1 + \mathring{\Delta}\underline{\theta}\Big) \\ &\quad + \nabla\cdot t + G_{\underline{L}}(\nabla\omega) - \frac{1}{2}\nabla\omega\operatorname{tr} K - 2\hat{K}(\vec{t},\nabla\omega) \\ &\quad + \frac{1}{2}(|\hat{K}|^2 + G(\underline{L},\underline{L}))|\nabla\omega|^2 \\ &\quad - \Delta\log\operatorname{tr} K - \nabla\omega\underline{L}\log\operatorname{tr} K + 2\hat{K}(\nabla\log\operatorname{tr} K, \nabla\omega)\Big)dA_{s_\star}\end{aligned}$$

$$+ o(s_\star^{-1})$$

having used the Divergence Theorem to get the second line. From Lemma 5.2.2 we have

$$-\Delta \log \operatorname{tr} K - \nabla\omega \underline{L} \log \operatorname{tr} K + 2\hat{K}(\nabla \log \operatorname{tr} K, \nabla\omega) = -\not{\Delta} \log \operatorname{tr} K + \not{\nabla} \cdot (\underline{L} \log \operatorname{tr} K \not{\nabla}\omega)$$

and therefore integrates to zero on $\Sigma_{s_\star}$ by the Divergence Theorem. We also notice from the fact that $\underline{L}$ is geodesic and $\not{\nabla}\omega = \nabla\omega + |\nabla\omega|^2 \underline{L}$ that

$$K(V, \nabla\omega) = K(\tilde{V}, \not{\nabla}\omega) = \underline{\chi}(\tilde{V}, \not{\nabla}\omega)$$

for $V \in E(\Sigma_0)$. Lemma 5.1.1 and 5.2.1 therefore gives

$$\begin{aligned}
&\not{\nabla}_{\tilde{V}} \zeta(\tilde{W}) + (\not{\nabla}_{\tilde{V}}(\underline{\chi} \cdot \not{d}\omega))(\tilde{W}) = \tilde{V}(\zeta(\tilde{W}) + \underline{\chi}(\tilde{W}, \not{\nabla}\omega)) - \zeta(\not{\nabla}_{\tilde{V}} \tilde{W}) - \underline{\chi}(\not{\nabla}_{\tilde{V}} \tilde{W}, \not{\nabla}\omega) \\
&= (V + V\omega \underline{L}) t(W) - \Big(t(\nabla_V W + V\omega \vec{K}(W) + W\omega \vec{K}(V) - K(V, W)\nabla\omega) \\
&\quad - K(\nabla_V W + V\omega \vec{K}(W) + W\omega \vec{K}(V) - K(V, W)\nabla\omega, \nabla\omega) \Big) - \underline{\chi}(\not{\nabla}_{\tilde{V}} \tilde{W}, \not{\nabla}\omega) \\
&= \nabla_V t(W) + V\omega \mathcal{L}_{\underline{L}} t(W) - V\omega K(\vec{t}, \nabla\omega) - W\omega K(\vec{t}, \nabla\omega) + K(V, W) t(\nabla\omega)
\end{aligned}$$

where all terms in the penultimate line canceled from Lemma 5.2.1. Taking a trace over V, W

$$\begin{aligned}
&\not{\nabla} \cdot \zeta - \not{\nabla} \cdot (\vec{\underline{\chi}}(\not{\nabla}\omega)) = \nabla \cdot t + \mathcal{L}_{\underline{L}} t(\nabla\omega) - 2\hat{K}(\vec{t}, \nabla\omega) \\
&= \nabla \cdot t + G_{\underline{L}}(\nabla\omega) - \frac{1}{2} \nabla\omega \operatorname{tr} K - 2\hat{K}(\vec{t}, \nabla\omega) - \nabla \cdot \hat{K}(\nabla\omega) + \nabla\omega \operatorname{tr} K - \operatorname{tr} K t(\nabla\omega)
\end{aligned}$$

having used (11) to get the last line. We conclude that

$$\int \nabla \cdot t + G_{\underline{L}}(\nabla\omega) - \frac{1}{2} \nabla\omega \operatorname{tr} K - 2\hat{K}(\vec{t}, \nabla\omega) dA_{s_\star} = \int \nabla \cdot \hat{K}(\nabla\omega) - \nabla\omega \operatorname{tr} K + \operatorname{tr} K t(\nabla\omega) dA_{s_\star}$$

giving

$$4\pi \frac{E_H(\Sigma_{s_\star})}{\sqrt{\frac{|\Sigma_{s_\star}|}{16\pi}}} = \int \Big(\frac{1}{\omega^3} \Big(\frac{1}{2} \mathring{\mathcal{K}} \underline{\theta} - \frac{1}{2} \theta + \frac{1}{2} \mathring{\nabla} \cdot \mathring{\nabla} \cdot \gamma_1 + \mathring{\Delta} \underline{\theta} \Big)$$

$$+ \nabla \cdot \hat{K}(\nabla\omega) - \nabla\omega \operatorname{tr} K + \operatorname{tr} K t(\nabla\omega) + \frac{1}{2}(|\hat{K}|^2 + G(\underline{L}, \underline{L}))|\nabla\omega|^2\Big) dA_{s_\star}$$

$$+ o(s_\star^{-1}).$$

Now we turn to simplifying the final term in the integrand

$$(|\hat{K}|^2 + G(\underline{L}, \underline{L}))|\nabla\omega|^2 = (\nabla\omega - \nabla\!\!\!/\omega) \operatorname{tr} K - \frac{1}{2}(\operatorname{tr} K)^2 |\nabla\omega|^2.$$

Denoting $g := \operatorname{tr} K - \frac{2}{s} - \frac{\theta}{s^2}$ we conclude from the hypothesis $\mathcal{L}_{\underline{L}}\tilde{\gamma} = o_1^X(1)$ that $g = o_1^X(s^{-2})$. So denoting $g_\omega := g|_{\Sigma_{s_\star}}$ we have from Remark 5.3.2 (regarding the decay) that

$$\begin{aligned}
\nabla\omega \operatorname{tr} K|_{\Sigma_{s_\star}} &= \nabla\omega g|_{\Sigma_{s_\star}} + \frac{1}{s^2}\nabla\omega\underline{\theta}|_{\Sigma_{s_\star}} \\
&= \frac{1}{\omega^2}\nabla\!\!\!/\omega\underline{\theta} + o(s_\star^{-3}) \\
\nabla\!\!\!/\omega \operatorname{tr} K &= \nabla\!\!\!/\omega g_\omega + \nabla\!\!\!/\omega(\frac{2}{\omega} + \frac{\theta}{\omega^2}) \\
&= \nabla\!\!\!/ \cdot (g_\omega \nabla\!\!\!/\omega) - \Delta\!\!\!/\omega g_\omega - \frac{2}{\omega^2}|\nabla\!\!\!/\omega|^2 + \nabla\!\!\!/\omega(\frac{\theta}{\omega^2}) \\
&= \nabla\!\!\!/ \cdot (g_\omega \nabla\!\!\!/\omega) - (\Delta\omega - 2\hat{K}(\nabla\omega, \nabla\omega))g_\omega - \frac{2}{\omega^2}|\nabla\!\!\!/\omega|^2 + \nabla\!\!\!/\omega(\frac{\theta}{\omega^2}) \\
&= \nabla\!\!\!/ \cdot (g_\omega \nabla\!\!\!/\omega) - \frac{2}{\omega^2}|\nabla\!\!\!/\omega|^2 + \nabla\!\!\!/\omega(\frac{\theta}{\omega^2}) + o(s_\star^{-3}) \\
\frac{1}{2}(\operatorname{tr} K)^2|\nabla\omega|^2|_{\Sigma_{s_\star}} &= \frac{1}{2}(\frac{2}{\omega} + \frac{\theta}{\omega^2})^2|\nabla\!\!\!/\omega|^2 + o(s_\star^{-3}) \\
&= \frac{2}{\omega^2}|\nabla\!\!\!/\omega|^2 + \frac{2\theta}{\omega^3}|\nabla\!\!\!/\omega|^2 + o(s_\star^{-3}).
\end{aligned}$$

Combining terms we conclude

$$(|\hat{K}|^2 + G(\underline{L}, \underline{L}))|\nabla\omega|^2\Big|_{\Sigma_{s_\star}} = -\nabla\!\!\!/ \cdot (g_\omega \nabla\!\!\!/\omega) + o(s_\star^{-3}).$$

It's a simple exercise to show $\hat{K} = -\frac{1}{2}(\gamma_1 + \underline{\theta}\mathring{\gamma}) + o_1^X(1)$, so for $d\omega|_{\Sigma_{s_\star}}$

$= (s_\star^2\phi^2)(-\frac{1}{s_\star}d\phi^{-1}|_{\Sigma_{s_\star}} + o(s_\star^{-1}))$ we have from Lemma 5.3.2

$$\nabla\cdot\hat{K}(\nabla\omega)|_{\Sigma_{s_\star}} = \frac{1}{s_\star^3}\frac{1}{2\phi^2}\mathring{\nabla}\cdot(\gamma_1+\underline{\theta}\mathring{\gamma})(\mathring{\nabla}\phi^{-1}) + o(s_\star^{-3})$$

$$\nabla\omega\,\mathrm{tr}\,K|_{\Sigma_{s_\star}} = -\frac{1}{s_\star^3}\frac{1}{\phi^2}\mathring{\nabla}\phi^{-1}\underline{\theta} + o(s_\star^{-3})$$

$$\mathrm{tr}\,Kt(\nabla\omega)|_{\Sigma_{s_\star}} = -\frac{1}{s_\star^3}\frac{2}{\phi^2}t_1(\mathring{\nabla}\phi^{-1}) + o(s_\star^{-3}).$$

Therefore

$$E_H(\Sigma_{s_\star}) = \frac{1}{8\pi}\sqrt{\frac{\int\phi^2 f d\mathring{A}}{4\pi}}\int\Big(\frac{f}{\phi}\Big(\frac{1}{2}\mathring{\mathcal{K}}\underline{\theta} + \frac{1}{2}\mathring{\nabla}\cdot\mathring{\nabla}\cdot\gamma_1 + \mathring{\Delta}\underline{\theta} - \frac{1}{2}\theta\Big)$$
$$+\frac{f}{2}\mathring{\nabla}\cdot(\gamma_1+\underline{\theta}\mathring{\gamma})(\mathring{\nabla}\phi^{-1}) + f\mathring{\nabla}\phi^{-1}\underline{\theta} - 2ft_1(\mathring{\nabla}\phi^{-1})\Big)d\mathring{A} + o(s_\star^0)$$

giving

$$\lim_{s_\star\to\infty} E_H(\Sigma_{s_\star}) = \frac{1}{8\pi}\sqrt{\frac{\int\phi^2 d\mathring{A}}{4\pi}}\int\Big(\frac{1}{\phi}\Big(\frac{1}{2}\mathring{\mathcal{K}}\underline{\theta} + \frac{1}{2}\mathring{\nabla}\cdot\mathring{\nabla}\cdot\gamma_1 + \mathring{\Delta}\underline{\theta} - \frac{1}{2}\theta\Big)$$
$$+\frac{1}{2}\mathring{\nabla}\cdot(\gamma_1+\underline{\theta}\mathring{\gamma})(\mathring{\nabla}\phi^{-1}) + \mathring{\nabla}\phi^{-1}\underline{\theta} - 2t_1(\mathring{\nabla}\phi^{-1})\Big)d\mathring{A}$$
$$= \frac{1}{8\pi}\sqrt{\frac{\int\phi^2 d\mathring{A}}{4\pi}}\int\Big(\frac{1}{\phi}\Big(\frac{1}{2}\mathring{\mathcal{K}}\underline{\theta} + \frac{1}{2}\mathring{\nabla}\cdot\mathring{\nabla}\cdot\gamma_1 + \mathring{\Delta}\underline{\theta} - \frac{1}{2}\theta\Big)$$
$$-\frac{1}{2}\phi^{-1}\mathring{\nabla}\cdot\mathring{\nabla}\cdot(\gamma_1+\underline{\theta}\mathring{\gamma}) - \phi^{-1}\mathring{\Delta}\underline{\theta} + \frac{2}{\phi}\mathring{\nabla}\cdot t_1\Big)d\mathring{A}$$

$$= \frac{1}{16\pi}\sqrt{\frac{\int\phi^2 d\mathring{A}}{4\pi}}\int\frac{1}{\phi}\Big(\mathring{\mathcal{K}}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla}\cdot t_1\Big)d\mathring{A}$$

having integrated by parts to get the second equality. □

Remark 5.3.3. *Suppose Ω is a past asymptotically flat null hypersurface with a background geodesic foliation $\{\Sigma_s\}$ approaching large spheres (i.e $\mathring{\gamma}$ is the round metric* *at infinity). Then for any other geodesic foliation of scale factor ψ it follows that the metric at infinity is $\psi^2\mathring{\gamma}$ (see [18], Section 4) approaching large spheres if and only if ψ solves the equation*

$$1 - \psi^2 = \mathring{\Delta} \log \psi. \tag{5.7}$$

Proposition 5.3.3 shows all asymptotically geodesic foliations $\{\Sigma_{s_\star}\}$ of the same scale factor ϕ share the limit

$$E(\phi) = \lim_{s_\star \to \infty} E_H(\Sigma_{s_\star})$$

which measures a Bondi energy $E_B(\psi)$ if ψ solves (5.7). The Bondi mass is therefore given by

$$m_B = \inf\{E_B(\psi) | 1 - \psi^2 = \mathring{\Delta} \log \psi\}.$$

Theorem 5.3.4. *Suppose Ω is a past asymptotically flat null hypersurface inside a spacetime satisfying the dominant energy condition. Then given the existence of an asymptotically geodesic (P)-foliation $\{\Sigma_{s_\star}\}$ approaching large spheres we have*

$$m(0) \leqslant E_B$$

for E_B the Bondi energy of Ω associated to $\{\Sigma_{s_\star}\}$. If equality is achieved on an (SP)-foliation then $E_B = m_B$ the Bondi mass of Ω. In the case that $\operatorname{tr}\chi|_{\Sigma_0} = 0$ we conclude instead with the weak Null Penrose inequality

$$\sqrt{\frac{|\Sigma_0|}{16\pi}} \leqslant E_B$$

where equality along an (SP)-foliation enforces that any foliation of Ω shares its data (γ, $\underline{\chi}$, $\operatorname{tr}\chi$ and ζ) with some foliation of the standard nullcone of Schwarzschild spacetime.

Proof. Since any asymptotically geodesic (P)-foliation has non-decreasing mass from Theorem 2.1.1 and $m(\Sigma_{s_\star}) \leqslant E_H(\Sigma_{s_\star})$ from Lemma 3.2.2, it follows from [18] (Theorem 1) that $m(\Sigma_{s_\star})$ converges since $E_H(\Sigma_{s_\star})$ does. Moreover,

$$\lim_{s_\star\to\infty} m(\Sigma_{s_\star}) \leqslant \lim_{s_\star\to\infty} E_H(\Sigma_{s_\star})$$

and from [18] (Corollary 3) it follows that $\lim_{s_\star\to\infty} E_H(\Sigma_{s_\star})$ is the Bondi energy associated to the abstract reference frame coupled to the foliation $\{\Sigma_{s_\star}\}$. Given the case of equality, Theorem 2.1.1 enforces that $m(0) = m(\Sigma_{s_\star})$ for all $s_\star$. So Theorem 4.3.2 applies and we conclude that $m(\Sigma) = \frac{1}{2}\Big(\frac{1}{4\pi}\int r_0^{\frac{2}{3}} dA_0\Big)^{\frac{3}{2}}$ (for some positive function r_0 on Σ_0 of area form $r_0^2 dA_0$) irrespective of the cross-section $\Sigma\subset\Omega$. This gives, according to Remark 5.3.3 and Lemma 3.2.2,

$$\lim_{s_\star\to\infty} m(\Sigma_{s_\star}) = \frac{1}{2}\Big(\frac{1}{4\pi}\int r_0^{\frac{2}{3}} dA_0\Big)^{\frac{3}{2}} = E_B \leqslant \inf_{\phi>0} E(\phi) \leqslant m_B.$$

Since $E_B \leqslant \inf E(\phi) \leqslant m_B \leqslant E_B$ all must be equal.

If $\operatorname{tr}\chi|_{\Sigma_0} = 0$ property (P) gives

$$0 \geqslant \not{\Delta} \log \not{\phi}|_{\Sigma_0}$$

and the maximum principle implies $\not{\phi}|_{\Sigma_0} = \mathcal{K} + \not{\nabla}\cdot\tau$ is constant. From the Gauss-Bonnet and Divergence Theorems we conclude that $\not{\phi}|_{\Sigma_0} = \frac{4\pi}{|\Sigma_0|}$ from which it follows that $m(0) = \sqrt{\frac{|\Sigma_0|}{16\pi}}$. Under this restriction Theorem 4.3.2 enforces that any foliation of Ω corresponds with a foliation of the standard nullcone in Schwarzschild with respect to the data γ, $\underline{\chi}$, $\operatorname{tr}\chi$ and ζ. □

From Proposition 5.3.3 and Lemma 3.2.2

$$\inf_{\phi>0} E(\phi) = \frac{1}{4}\Big(\frac{1}{4\pi}\int (\mathcal{K}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla}\cdot t_1)^{\frac{2}{3}} d\mathring{A}\Big)^{\frac{3}{2}}$$

provided $\mathcal{K}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla}\cdot t_1 \geqslant 0$. We show, given that Ω satisfies the strong flux decay condition, this quantity is infact $\lim\limits_{s_\star\to\infty} m(\Sigma_{s_\star})$. We will need the following proposition to do so:

Proposition 5.3.5. *Suppose Ω is past asymptotically flat with strong decay. Given a choice of affinely parametrized null generator $\underline{L}$ and corresponding level set function, s, we have*

$$\nabla \cdot \nabla \cdot K = -\frac{1}{2s^4}\mathring{\nabla} \cdot \mathring{\nabla} \cdot \gamma_1 + o(s^{-4}) \tag{5.8}$$

$$\Delta \operatorname{tr} K = \frac{\mathring{\Delta}\underline{\theta}}{s^4} + o(s^{-4}) \tag{5.9}$$

$$\nabla \cdot t = \frac{1}{s^3}\mathring{\nabla} \cdot t_1 + o(s^{-3}) \tag{5.10}$$

Proof. From Lemma 5.3.2 and (5.2)

$$\begin{aligned}
\nabla_i K_{jm} &= \mathring{\nabla}_i(s\mathring{\gamma}_{jm} + \frac{1}{2}\gamma_{1jm}) - \mathcal{D}^k_{ij}K_{km} - \mathcal{D}^k_{im}K_{jk} + o^X_1(1) \\
&= \frac{1}{2}\mathring{\nabla}_i\gamma_{1jm} - \mathring{\nabla}_i\gamma_{1jm} + o^X_1(1) \\
&= -\frac{1}{2}\mathring{\nabla}_i\gamma_{1jm} + o^X_1(1).
\end{aligned}$$

where the first term of the second line comes from the fact that $\mathring{\nabla}\mathring{\gamma} = 0$. Next we compute

$$\begin{aligned}
\nabla_i\nabla_j K_{mn} &= \mathring{\nabla}_i\nabla_j K_{mn} - \mathcal{D}^k_{ij}\nabla_k K_{mn} - \mathcal{D}^k_{im}\nabla_j K_{kn} - \mathcal{D}^k_{in}\nabla_j K_{mk} \\
&= -\frac{1}{2}\mathring{\nabla}_i\mathring{\nabla}_j\gamma_{1mn} + o(1)
\end{aligned}$$

So contracting with (5.1) over j, m followed by i, n we get (5.8) and contracting instead over m, n and then i, j (5.9) follows. For (5.10)

$$\begin{aligned}
\nabla_i t_j &= \mathring{\nabla}_i t_j - \mathcal{D}^k_{ij}t_k \\
&= \frac{1}{s}\mathring{\nabla}_i t_{1j} + o(s^{-1})
\end{aligned}$$

and the result follows as soon as we contract with (5.1) over i, j. □

Remark 5.3.4. *As soon as we impose that Ω has strong decay it follows from the fact that $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$ that $\mathcal{L}_{X_i}\tilde{\gamma}$, and $\mathcal{L}_{X_i}\mathcal{L}_{X_j}\tilde{\gamma} = o_1(s)$, since, for example,*

$$\mathcal{L}_{\underline{L}}\mathcal{L}_{X_i}\tilde{\gamma} = \mathcal{L}_{[\underline{L},X_i]}\tilde{\gamma} + \mathcal{L}_{X_i}\mathcal{L}_{\underline{L}}\tilde{\gamma} = o(1).$$

As a result its not hard to see early in the proof of Proposition 7 that

$$\mathcal{K}_{\gamma(s)} = \frac{\mathring{\mathcal{K}}}{s^2} + \frac{\mathcal{K}_1}{s^3} + O_1(s^{-4})$$

for some Lie constant function $\mathcal{K}_1$. We may therefore provide a simpler proof using Proposition 5.3.5 and the propogation equation (4.1)

$$\underline{L}\mathcal{K} = -\operatorname{tr} K\mathcal{K} - \Delta \operatorname{tr} K + \nabla \cdot \nabla \cdot K$$

in order to find $\mathcal{K}_1$.

Theorem 5.3.6. *Suppose Ω is past asymptotically flat with strong flux decay and $\{\Sigma_s\}$ is some background geodesic foliation. Then for any asymptotically geodesic foliation $\{\Sigma_{s_\star}\}$ with scale factor $\phi > 0$ we have*

$$s_\star^3 \not{\rho}(s_\star) = \frac{1}{2\phi^3}\Big(\mathring{\mathcal{K}}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla} \cdot t_1\Big) + o(s_\star^0)$$

Proof. First let us remind ourselves of Theorem 5.2.1

$$\not{\rho} = \rho + \not{\nabla} \cdot \Big(\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr} K}\not{\nabla}\omega\Big) + \frac{1}{2}\Big(|\hat{K}|^2 + G(\underline{L},\underline{L})\Big)|\nabla\omega|^2$$
$$+ \nabla\omega\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr} K} + G_{\underline{L}}(\nabla\omega) - 2\hat{K}(\vec{t} - \nabla \log \operatorname{tr} K, \nabla\omega).$$

Denoting the exterior derivative on Σ_s by d_s, since $\operatorname{tr} K = \frac{2}{s} + \frac{\theta}{s^2} + o(s^{-2})$, we conclude that $d_s \log \operatorname{tr} K = \frac{1}{2s}d\underline{\theta}|_{\Sigma_s} + o(s^{-1})$ giving

$$\hat{K}(\vec{t} - \nabla \log \operatorname{tr} K, \nabla\omega)|_{\Sigma_{s_\star}} = o(s_\star^{-3}).$$

Since $\mathcal{L}^2_{\underline{L}}\tilde{\gamma} = o^X_1(s^{-1}) \cap o_1(s^{-1})$ we also see that

$$|\hat{K}|^2 + G(\underline{L}, \underline{L}) = -\underline{L}\operatorname{tr} K - \frac{1}{2}(\operatorname{tr} K)^2 = -(-\frac{2}{s^2} - \frac{2}{s^3}\underline{\theta}) - \frac{1}{2}(\frac{2}{s} + \frac{\theta}{s^2})^2 + o^X_1(s^{-3}) \cap o_1(s^{-3}) = o^X_1(s^{-3}) \cap o_1(s^{-3})$$

and therefore, from Remark 5.3.2:

$$\begin{aligned}\nabla\!\!\!/ \cdot \Big(\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K}\nabla\!\!\!/\omega\Big) &= \nabla\!\!\!/\omega\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K} + \Delta\!\!\!/\omega\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K} \\ &= \Big(\nabla\omega\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K} + |\nabla\omega|^2\underline{L}\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K} \\ &\quad + (\Delta\omega - 2\hat{K}(\nabla\omega, \nabla\omega))\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K}\Big)\Big|_{\Sigma_{s_\star}} \\ &= o(s_\star^{-3}).\end{aligned}$$

From the strong flux decay condition we have $G_{\underline{L}}(\nabla\omega)|_{\Sigma_{s_\star}} = o(s_\star^{-3})$ also. From (5.9) we have

$$\Delta \log \operatorname{tr} K = \frac{\Delta \operatorname{tr} K}{\operatorname{tr} K} - \frac{|\nabla \operatorname{tr} K|^2}{(\operatorname{tr} K)^2} = \frac{\mathring{\Delta}\underline{\theta}}{2s^3} + o(s^{-3})$$

and combining this with Propositions 5.3.2 and 5.3.5:

$$\begin{aligned}\rho\!\!\!/ &= \rho|_{\Sigma_{s_\star}} + o(s_\star^{-3}) \\ &= \frac{1}{\omega^3}\Big(\frac{1}{2}\mathring{\mathcal{K}}\underline{\theta} + \frac{1}{2}\mathring{\nabla}\cdot\mathring{\nabla}\cdot\gamma_1 + \mathring{\Delta}\underline{\theta} - \frac{1}{2}\theta\Big) + \frac{1}{\omega^3}\mathring{\nabla}\cdot t_1 - \frac{1}{2\omega^3}\mathring{\Delta}\underline{\theta} + o(s_\star^{-3}) \\ &= \frac{1}{2\omega^3}\Big(\mathring{\mathcal{K}}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla}\cdot t_1\Big) + o(s_\star^{-3})\end{aligned}$$

having used Proposition 5.3.1 in the final line to substitute $\frac{1}{2}\mathring{\nabla}\cdot\mathring{\nabla}\cdot\gamma_1 + \mathring{\Delta}\underline{\theta} = \mathring{\nabla}\cdot t_1$ and the result follows. □

Remark 5.3.5. *We would like to bring to the attention of the reader our use of (5.4) in the second to last equality in the proof of Theorem 5.3.6. Assuming $\{\Sigma_{s_\star}\}$ is infact a geodesic foliation, running a parallel argument to decompose $\not{\rho}$ as we did for ρ allows us to conclude that (5.4) must also hold for $\{\Sigma_{s_\star}\}$. We refer the reader to [18] (Proposition 3) to observe that under the additional decay of Theorem 5.3.6, part 4 from Definition 5.3.1 is no longer necessary to give (5.4) for an arbitrary geodesic foliation provided it holds for at least one. We will exploit this fact in Section 6.2.*

Corollary 5.3.6.1. *With the same hypotheses as in Theorem 5.3.6 we have*

$$\lim_{s_\star\to\infty} m(\Sigma_{s_\star}) = \frac{1}{4}\Big(\frac{1}{4\pi}\int(\mathring{\mathcal{K}}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla}\cdot t_1)^{\frac{2}{3}} \mathring{dA}\Big)^{\frac{3}{2}}$$

Proof. From Theorem 5.3.6 we directly conclude

$$4\pi(4m(\Sigma_{s_\star}))^{\frac{2}{3}} = \int (2\not{\rho})^{\frac{2}{3}} dA_\omega = \int \frac{1}{\omega^2}\Big(\mathring{\mathcal{K}}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla}\cdot t_1 + o(1)\Big)^{\frac{2}{3}} f\omega^2 \mathring{dA}$$

giving

$$4\pi(4\lim_{s_\star\to\infty} m(\Sigma_{s_\star}))^{\frac{2}{3}} = \int \Big(\mathring{\mathcal{K}}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla}\cdot t_1\Big)^{\frac{2}{3}} \mathring{dA}$$

by the Dominated Convergence Theorem. □

Finally we're ready to prove Theorem 2.1.2:

Proof. (Theorem 2.1.2) The first claim of Theorem 2.1.2 is a simple consequence of Theorem 2.1.1. Property (P) and Theorem 5.3.6 enforces that

$$0 \leqslant \lim_{s_\star\to\infty} s_\star^3 \not{\rho} = \frac{1}{2\phi^3}(\mathring{\mathcal{K}}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla}\cdot t_1)$$

therefore, Theorem 2.1.1, Corollary 5.3.6.1, Lemma 3.2.2 and Proposition 5.3.3 gives

$$m(\Sigma_0) \leqslant \lim_{s_\star\to\infty} m(\Sigma_{s_\star}) = \frac{1}{4}\Big(\frac{1}{4\pi}\int(\mathring{\mathcal{K}}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla}\cdot t_1)^{\frac{2}{3}} \mathring{dA}\Big)^{\frac{3}{2}} = \inf_{\phi>0} E(\phi) \leqslant m_B.$$

The rest of the proof is settled identically as in Theorem 5.3.4. □

6

Perturbing Spherical Symmetry

6.1 Spherical Symmetry

For the known null Penrose inequality in spherical symmetry (see [14]) we provide proof within our context in order to motivate a class of perturbations on the black hole exterior that maintain both the asymptotically flat and strong flux decay conditions. We also show the existence of an asymptotically geodesic (SP)-foliation for a subclass of these perturbations toward a proof of the null Penrose conjecture.

6.1.1 The metric

In polar areal coordinates [22] the metric takes the form

$$g = -a(t,r)^2 dt \otimes dt + b(t,r)^2 dr \otimes dr + r^2 \mathring{\gamma}$$

for $\mathring{\gamma}$ the standard round metric on $\mathbb{S}^2$. From which the change in coordinates $(t,r) \to (v,r)$ given by

$$dv = dt + \frac{b}{a} dr$$

produces the metric and metric inverse given by

$$g = -\mathfrak{h} e^{2\beta} dv \otimes dv + e^{\beta}(dv \otimes dr + dr \otimes dv) + r^2 \mathring{\gamma}$$

$$g^{-1} = e^{-\beta}(\partial_v \otimes \partial_r + \partial_r \otimes \partial_v) + \mathfrak{h}\partial_r \otimes \partial_r + \frac{1}{r^2}\mathring{\gamma}^{-1}$$

for $\mathfrak{h} = (1 - \frac{2M(t,r)}{r})$ where $M(t,r) := \frac{r}{2}(1 - \frac{1}{b^2})$ and $a(t,r)^2 = \mathfrak{h}e^{2\beta}$.

It's a well known fact that assigning $M(t,r) = m_0 > 0$ and $\beta(t,r) = 0$ for m_0 a constant the above metric covers the region given by $v > 0$ in Kruskal spacetime or Schwarzschild geometry in an 'Eddington-Finkelstein' coordinate chart. We will therefore refer to the null hypersurfaces $\Omega := \{v = v_0\}$ as the *standard nullcones* (of spherically symmetric spacetime) as they agree with the similarly named hypersurfaces in the Schwarzschild case.

6.1.2 Calculating ρ

We approach the calculation similarly to the case of Schwarzschild. Denoting the gradient of v by Dv we use the identity $D_{Dv}Dv = \frac{1}{2}D|Dv|^2$ to see $\underline{L} := Dv = e^{-\beta}\partial_r$ satisfies $D_{\underline{L}}\underline{L} = 0$ providing us our choice of geodesic generator for Ω and level set function s (as in Section 4.1). For convenience we will choose our background foliation $\{\Sigma_r\}$ of Ω to be the level sets of the coordinate r. An arbitrary cross-section Σ of Ω is therefore given as a graph over Σ_{r_0} (for some r_0) which we Lie drag along ∂_r to the rest of Ω giving some $\omega \in \mathcal{F}(\Omega)$. On Σ we therefore have the linearly independent normal vector fields

$$\underline{L} = e^{-\beta}\partial_r$$
$$D(r - \omega) = e^{-\beta}\partial_v + \mathfrak{h}\partial_r - \nabla\omega$$

where in this subsection (6.1.2) ∇ will temporarily denote the induced covariant derivative on Σ_r. We wish to find the null section $L \in \Gamma(T^{\perp}\Sigma)$ satisfying $\langle L, \underline{L}\rangle = 2$. Since $L = c_1\underline{L} + c_2 D(r - \omega)$ we have

$$\begin{aligned} 2 = \langle L, \underline{L}\rangle &= c_2 e^{-\beta}\partial_r(r - \omega) \\ &= c_2 e^{-\beta} \end{aligned}$$

$$
\begin{aligned}
0 = \langle L, L\rangle &= 2c_1c_2\langle \underline{L}, D(r-\omega)\rangle + c_2^2\langle D(r-\omega), D(r-\omega)\rangle \\
&= 2c_1c_2e^{-\beta} + c_2^2(e^{-2\beta}\langle \partial_v, \partial_v\rangle + |\nabla\omega|^2 + 2e^{-\beta}\mathfrak{h}\langle \partial_v, \partial_r\rangle) \\
&= 2c_1c_2e^{-\beta} + c_2^2(\mathfrak{h} + |\nabla\omega|^2)
\end{aligned}
$$

giving $c_2 = 2e^{\beta}$ and $c_1 = -e^{2\beta}(\mathfrak{h} + |\nabla\omega|^2)$ so that

$$
\begin{aligned}
L &= -e^{\beta}(\mathfrak{h} + |\nabla\omega|^2)\partial_r + 2\partial_v + 2\mathfrak{h}e^{\beta}\partial_r - 2e^{\beta}\nabla\omega \\
&= 2\partial_v + e^{\beta}(\mathfrak{h} - |\nabla\omega|^2)\partial_r - 2e^{\beta}\nabla\omega \\
&= 2\partial_v + e^{\beta}(\mathfrak{h} - |\not\nabla\omega|^2)\partial_r - 2e^{\beta}(\not\nabla\omega - |\not\nabla\omega|^2\partial_r) \\
&= 2\partial_v + e^{\beta}(\mathfrak{h} + |\not\nabla\omega|^2)\partial_r - 2e^{\beta}\not\nabla\omega
\end{aligned}
$$

having used the fact that $\not\nabla\omega = \nabla\omega + |\nabla\omega|^2\partial_r$ to get the third equality. We note from the warped product structure (as for Kruskal spacetime) that $E_{\partial_r}(\Sigma_{r_0}) = \mathcal{L}(\mathbb{S}^2)|_{\Omega}$ where $\mathcal{L}(\mathbb{S}^2)$ is the set of lifted vector fields from the $\mathbb{S}^2$ factor of the spacetime product manifold. As a result we may globally extend $V \in E_{\partial_r}(\Sigma_{r_0})$ to satisfy $[\partial_v, V] = 0$. The following facts are a direct application of the Koszul formula, we refer the reader to [21] (pg.206) for the details:

$$D_{\partial_r}\partial_v = -\frac{1}{2}\partial_r(\mathfrak{h}e^{2\beta})e^{-\beta}\partial_r \tag{6.1}$$

$$D_V\partial_v = 0 \tag{6.2}$$

$$D_{\partial_r}\partial_r = \partial_r\beta\partial_r \tag{6.3}$$

$$D_V\partial_r = \frac{1}{r}V. \tag{6.4}$$

Lemma 6.1.1. *Suppose $\Omega = \{v = v_0\}$ is the standard null cone in a spherically symmetric spacetime of metric*

$$g = -\mathfrak{h}e^{2\beta(v,r)}dv \otimes dv + e^{\beta}(v,r)(dv \otimes dr + dr \otimes dv) + r^2\mathring{\gamma}$$

where $\mathfrak{h} = (1 - \frac{2M(v,r)}{r})$ and $\mathring{\gamma}$ is the round metric on $\mathbb{S}^2$. Then for some cross-section $\Sigma_{r_0} \subset \Omega$ and $\omega \in \mathcal{F}(\Sigma_{r_0})$, $\Sigma := \{r = \omega \circ \pi\}$ produces the data (writing $\omega \circ \pi$ as ω):

$$\gamma = \omega^2\mathring{\gamma}$$

$$\underline{\chi} = \frac{e^{-\beta(v_0,\omega)}}{\omega}\gamma$$

$$\operatorname{tr}\underline{\chi} = \frac{2e^{-\beta(v_0,\omega)}}{\omega}$$

$$\chi = e^{\beta(v_0,\omega)}\Big((\mathfrak{h} + |\nabla\!\!\!/\omega|^2)\frac{\gamma}{\omega} - 2\tilde{H}^\omega - 2\beta_r d\!\!\!/\omega \otimes d\!\!\!/\omega\Big)$$

$$\operatorname{tr}\chi = \frac{2e^\beta(v_0,\omega)}{\omega}(\mathfrak{h} - \omega^2 \Delta\!\!\!/ \log\omega - \omega\beta_r|\nabla\!\!\!/\omega|^2)$$

$$\zeta = -d\!\!\!/ \log\omega$$

$$\rho = \frac{2M(v_0,\omega)}{\omega^3} + \Delta\!\!\!/\omega + \frac{\beta_r}{\omega}|\nabla\!\!\!/\omega|^2$$

Proof. For any $V \in E_{\partial_r}(\Sigma_{r_0})$ we have from Lemma 5.1.1 that $\tilde{V} := V + V\omega\partial_r|_\Sigma \in \Gamma(T\Sigma)$ so that the first identity follows directly from the metric restriction. From (25):

$$D_{\tilde{V}}\underline{L} = e^{-\beta}D_V(\partial_r) + e^\beta V\omega D_{\underline{L}}\underline{L} = \frac{e^{-\beta}}{r}V$$

so the second identity is given by

$$\begin{aligned}\underline{\chi}(\tilde{V},\tilde{W}) &= \langle D_{\tilde{V}}\underline{L}, \tilde{W}\rangle \\ &= \frac{e^{-\beta}}{r}\langle V, W\rangle\end{aligned}$$

and a trace over V, W gives the third so that $\Delta\!\!\!/ \log\operatorname{tr}\underline{\chi} = -\Delta\!\!\!/\beta - \Delta\!\!\!/ \log\omega$. For the forth identity:

$$\begin{aligned}\chi(\tilde{V},\tilde{W}) &= 2\langle D_{\tilde{V}}\partial_v, \tilde{W}\rangle + e^\beta(\mathfrak{h} + |\nabla\!\!\!/\omega|^2)\langle D_{\tilde{V}}\partial_r, \tilde{W}\rangle - 2e^\beta\langle D_{\tilde{V}}\nabla\!\!\!/\omega, \tilde{W}\rangle - 2\beta_r e^\beta \tilde{V}\omega\tilde{W}\omega \\ &= e^\beta(\mathfrak{h} + |\nabla\!\!\!/\omega|^2)\frac{1}{\omega}\langle\tilde{V},\tilde{W}\rangle - 2e^\beta\tilde{H}^\omega(\tilde{V},\tilde{W}) - 2\beta_r e^\beta(d\!\!\!/\omega \otimes d\!\!\!/\omega)(\tilde{V},\tilde{W})\end{aligned}$$

where $\langle D_{\tilde{V}}\partial_v, \tilde{W}\rangle = 0$ from (6.1) and (6.2) to give the second equality. Taking a trace over $\tilde{V}, \tilde{W}$ we conclude with the fifth identity:

$$\operatorname{tr}\chi|_\Sigma = \frac{2e^{\beta(v_0,\omega)}}{\omega}(\mathfrak{h} + |\nabla\!\!\!/\omega|^2 - \omega\Delta\!\!\!/\omega) - 2\beta_r e^{\beta(v_0,\omega)}|\nabla\!\!\!/\omega|^2$$

$$
\begin{aligned}
&= \frac{2e^{\beta}}{\omega}(\mathfrak{h} - \omega^2(\frac{\not{\Delta}\omega}{\omega} - \frac{|\not{\nabla}\omega|^2}{\omega^2})) - 2\beta_r e^{\beta}|\not{\nabla}\omega|^2 \\
&= \frac{2e^{\beta}}{\omega}(\mathfrak{h} - \omega^2\not{\Delta}\log\omega - \omega\beta_r|\not{\nabla}\omega|^2).
\end{aligned}
$$

As a result we have that

$$
\langle \vec{H}, \vec{H} \rangle = \operatorname{tr}\underline{\chi}\operatorname{tr}\chi = \frac{4}{\omega^2}(\mathfrak{h} - \omega^2\not{\Delta}\log\omega - \omega\beta_r|\not{\nabla}\omega|^2).
$$

Since the metric on Σ is given by $\omega^2\mathring{\gamma}$ we conclude that it has Gaussian curvature

$$
\mathcal{K} = \frac{1}{\omega^2}(1 - \mathring{\Delta}\log\omega) = \frac{1}{\omega^2} - \not{\Delta}\log\omega
$$

and therefore

$$
\mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H} \rangle = \frac{2M(v_0, \omega)}{\omega^3} + \frac{\beta_r}{\omega}|\not{\nabla}\omega|^2.
$$

Moreover, the torsion is given by

$$
\begin{aligned}
\zeta(\tilde{V}) &= \frac{1}{2}\langle D_{\tilde{V}}\underline{L}, L \rangle \\
&= \frac{e^{-\beta}}{2r}\langle V, L \rangle \\
&= -\frac{1}{r}V\omega \\
&= -\frac{1}{r}\tilde{V}\omega
\end{aligned}
$$

from which we conclude $\zeta(\tilde{V})|_{\Sigma} = -\tilde{V}\log\omega$ and $\not{\nabla}\cdot\zeta = -\not{\Delta}\log\omega$, giving

$$
\rho = \frac{2M(v_0, \omega)}{\omega^3} + \not{\Delta}\beta + \frac{\beta_r}{\omega}|\not{\nabla}\omega|^2.
$$

□

Remark 6.1.1. *We recover the data of Lemma 3.2.1 as soon as we set $m_0 = M$, $\beta = 0$ and $r_0 = 2m_0$ as expected.*

In comparison to the Schwarzschild spacetime, we have the two additional terms $\not\Delta\beta + \frac{\beta_r}{\omega}|\not\nabla\omega|^2$ in the flux function ρ. It turns out that a non-trivial $G(\underline{L},\underline{L})$ is responsible. Since $\not\Delta\beta = \beta_{rr}|\not\nabla\omega|^2 + \beta_r\not\Delta\omega$ and

$$\begin{aligned} G(\underline{L},\underline{L}) &= -\underline{L}\,\mathrm{tr}\,K - \frac{1}{2}(\mathrm{tr}\,K)^2 - |\hat{K}|^2 \\ &= -e^{-\beta}\partial_r(\frac{2e^{-\beta}}{r}) - \frac{1}{2}\frac{4e^{-2\beta}}{r^2} \\ &= \frac{2\beta_r}{r}e^{-2\beta} \end{aligned}$$

it follows, for arbitrary ω, that $\not\Delta\beta(\omega) + \frac{\beta_r}{\omega}|\not\nabla\omega|^2 = 0$ if and only if β is independent of the r-coordinate and therefore $G(\underline{L},\underline{L}) = 0$. For the function $M(v_0, r)$ we look to $G(\underline{L}, L)$ along the foliation $\{\Sigma_r\}$ since:

$$\begin{aligned} G(\underline{L},L) &= \underline{L}\,\mathrm{tr}\,\chi - 2\mathcal{K}_s + 2\nabla\cdot t + 2|\vec{t}|^2 + \langle\vec{H},\vec{H}\rangle \\ &= e^{-\beta}\partial_r(\frac{2e^{\beta}}{r}(1-\frac{2M}{r})) - \frac{2}{r^2} + \frac{4}{r^2}(1-\frac{2M}{r}) \\ &= \frac{2\beta_r}{r}(1-\frac{2M}{r}) - \frac{4M_r}{r^2}. \end{aligned}$$

It follows from Lemma 4.3.1, on Σ_r, that

$$G_{\underline{L}} = 0.$$

Since these components are all that contribute to the monotonicity of (2.2) for the foliation $\{\Sigma_r\}$ we see that our need of the dominant energy condition reduces to

$$0 \leqslant \mathfrak{h}\beta_r \leqslant \frac{2M_r}{r}$$

on $\{\mathfrak{h} \geqslant 0\} \cap \Omega$. Next we show that $\{\Sigma_r\}$ is a re-parametrization of a geodesic (SP)-foliation:

6.1.3 Asymptotic flatness

We now wish to choose the necessary decay on β and M in order to employ Theorem 2.1.2. For $\underline{L} = e^{-\beta}\partial_r$ the geodesic foliation $\{\Sigma_s\}$ has level set function given by

$$s(r) = \int_{r_0}^{r} e^{\beta(t)}dt$$

for which $\omega = const. \iff s = const.$ and therefore

$$\rho(s) = \frac{2M(r(s))}{r(s)^3}.$$

It follows from Lemma 6.1.1 that $\frac{1}{4}\langle\vec{H},\vec{H}\rangle - \frac{1}{3}\Delta\log\rho = \frac{\mathfrak{h}}{r(s)^2} > 0 \iff r(s) > r_0 = 2M(v_0, r_0)$ as in Schwarzschild.

Lemma 6.1.2. *Choosing $|\beta(v_0, r)| = o_2(r^{-1})$ integrable and $M(v_0, r) = m_0 + o(r^0)$ for some constant m_0, Ω is asymptotically flat with strong flux decay.*

Proof. We've already verified that $G_{\underline{L}} = 0$. Since $\frac{ds}{dr} = e^{\beta(r)} = (1 + \beta\frac{e^\beta - 1}{\beta})$, $|\beta|$ is integrable and $\frac{e^\beta-1}{\beta}$ is bounded it follows that $\frac{ds}{dr} = 1 + f$ where $|f| = o_2(r^{-1})$ is integrable. As a result

$$s = r - r_0 + \int_{r_0}^{\infty} f(t)dt - \int_{r}^{\infty} f(t)dt = r - c_0 + o_3(r^0)$$

where $\beta_0 = \int_{r_0}^{\infty} f(t)dt$ and $c_0 = r_0 - \beta_0$. We conclude that $r(s) = s + c_0 + o_3(s^0)$ since our assumptions on β imply that $\int_{r(s)}^{\infty} f(t)dt = o_3(s^0)$. From the fact that

$$\gamma_s = r^2\mathring{\gamma}|_{\Sigma_s} = (s + c_0 + o_3(1))^2\mathring{\gamma} = s^2(1 + \frac{c_0}{s} + o_3(s^{-1}))^2\mathring{\gamma} = s^2\mathring{\gamma} + 2c_0 s\mathring{\gamma} + o_3(s)\mathring{\gamma}$$

we see $\tilde{\gamma} = o_3(s)\mathring{\gamma}$ ensuring condition 1 of Definition 5.3.1 holds up to strong decay given that all dependence on tangential derivatives falls on the $\underline{L}$ Lie constant tensor

$\mathring{\gamma}$. Since $\vec{t} = 0$ for this foliation condition 2 follows trivially up to strong decay. If we assume that $M(v_0, r) = m_0 + o(1)$ for some constant m_0 we see directly from Lemma 6.1.1

$$
\begin{aligned}
\operatorname{tr} Q &= \operatorname{tr} \chi|_{\Sigma_s} \\
&= \frac{2}{r}(1 - \frac{2M}{r})|_{\Sigma_s} + o(s^{-2}) \\
&= \frac{2}{s}(1 - \frac{c_0}{s})(1 - \frac{2m_0}{s}) + o(s^{-2}) \\
&= \frac{2}{s} - 2\frac{c_0 + 2m_0}{s^2} + o(s^{-2})
\end{aligned}
$$

giving us the third condition of Definition 5.3.1.
We refer the reader to [18] to observe the use of the forth condition of Definition 5.3.1 in proving (5.4) for an arbitrary geodesic foliation. As mentioned in Remark 5.3.5, strong flux decay bypasses our need of this condition since $\operatorname{tr} Q = \frac{2\mathring{\mathcal{K}}}{s} + o(s^{-1})$ is verified above. □

From Lemma 6.1.2, Theorem 2.1.2, Theorem 4.3.2 and the comments immediately proceeding Remark 6.1.1 we have the following proof of the known (see [14]) null Penrose conjecture in spherical symmetry:

Theorem 6.1.1. *Suppose $\Omega := \{v = v_0\}$ is a standard null cone of a spherically symmetric spacetime of metric*

$$ds^2 = -\Big(1 - \frac{2M(v,r)}{r}\Big)e^{2\beta(v,r)}dv^2 + 2e^{\beta(v,r)}dvdr + r^2\Big(d\vartheta^2 + \sin\vartheta^2 d\varphi^2\Big)$$

where

1. $|\beta(v_0, r)| = o_2(r^{-1})$ *is integrable*

2. $M(v_0, r) = m_0 + o(r^0)$ *for some constant* $m_0 > 0$

3. $0 \leqslant \mathfrak{h}\beta_r \leqslant \frac{2M_r}{r}$

Then,

$$\sqrt{\frac{|\Sigma|}{16\pi}} \leqslant m_0$$

for m_0 the Bondi mass of Ω and $\Sigma := \{r_0 = 2M(v_0, r_0)\}$. In the case of equality we have $\beta = 0$ and $M = m_0$ so that Ω is a standard null cone of Schwarzschild spacetime.

6.2 Perturbing Spherical Symmetry

We wish to study perturbations off of the spherically symmetric metric given in Theorem 6.1.1 for the coordinate chart $(v, r, \vartheta, \varphi)$. We start by choosing a 1-form η such that $\eta(\partial_r(\partial_v)) = \mathcal{L}_{\partial_v}\eta = 0$ and a 2-tensor γ satisfying $\gamma(\partial_r(\partial_v), \cdot) = \mathcal{L}_{\partial_v}\gamma = 0$ with restriction $\gamma|_{(v,r)\times\mathbb{S}^2}$ positive definite. Finally we choose smooth functions M, β and α. Defining $\vec{\eta}$ to be the unique vector field satisfying $\gamma(\vec{\eta}, X) = \eta(X)$ for arbitrary $X \in \Gamma(TM)$ and $r^2|\vec{\eta}|^2 := \gamma(\vec{\eta}, \vec{\eta})$ the spacetime metric and its inverse are given by

$$g = -(\mathfrak{h} + \alpha)e^{2\beta}dv \otimes dv + e^{\beta}(dv \otimes (dr + \eta) + (dr + \eta) \otimes dv) + r^2\gamma$$

$$g^{-1} = e^{-\beta}(\partial_v \otimes \partial_r + \partial_r \otimes \partial_v) + (\mathfrak{h} + \alpha + |\vec{\eta}|^2)\partial_r \otimes \partial_r - (\vec{\eta} \otimes \partial_r + \partial_r \otimes \vec{\eta}) + \frac{1}{r^2}\gamma^{-1}.$$

We see that $\Omega := \{v = v_0\}$ remains a null hypersurface with $\underline{L}(= Dv) = e^{-\beta}\partial_r \in \Gamma(T\Omega) \cap \Gamma(T^{\perp}\Omega)$. Our metric resembles the perturbed metric used by Alexakis [1] to successfully verify the Penrose inequality for vacuum perturbations of the standard null cone of Schwarzschild spacetime. We'll need the following to specify our decay conditions:

Definition 6.2.1. *Suppose Ω extends to past null infinity with level set function, s, for some null generator $\underline{L}$. For a transversal k-tensor T*

- *We say* $T(s,\delta) = \delta o_n^X(s^{-m})$ *if* $T = o_n^X(s^{-m})$ *and*

$$\limsup_{\delta\to 0}\sup_{\Omega}\frac{1}{\delta}|s^m(\mathcal{L}_{X_{i_1}}...\mathcal{L}_{X_{i_j}}T)(s,\delta)| < \infty \text{ for } 0 \leqslant j \leqslant n$$

- *We define*

$$|T|^2_{\mathring{H}^m} = |T|^2_{\mathring{\gamma}} + |\mathring{\nabla}T|^2_{\mathring{\gamma}} + \cdots + |\mathring{\nabla}^m T|^2_{\mathring{\gamma}}.$$

Decay Conditions on Ω:

1. $r^2\gamma = r^2\mathring{\gamma} + r\delta\gamma_1 + \tilde{\gamma}$ where:
   (a) $\mathring{\gamma}$ is the ∂_r-Lie constant, transversal standard round metric on $\mathbb{S}^2$ independent of δ
   (b) γ_1 is a ∂_r-Lie constant, transversal 2-tensor independent of δ
   (c) $\tilde{\gamma}$ is a transversal 2-tensor satisfying $(\mathcal{L}_{\partial_r})^i\tilde{\gamma} = \delta o^X_{5-i}(r^{1-i})$ for $0 \leqslant i \leqslant 3$
2. $\alpha = \delta\frac{\alpha_0}{r} + \tilde{\alpha}$ where α_0 is a ∂_r-Lie constant function independent of δ and $|\tilde{\alpha}|_{\mathring{H}^2} \leqslant \delta h_1(r)$ for $h_1 = o(r^{-1})$
3. β satisfies:
   (a) $|\beta| = o_2(r^{-1})$ is r-integrable
   (b) $|\mathring{\nabla}\beta|_{\mathring{H}^3} \leqslant \delta h_2(r)$ for some integrable $h_2 = o(r^{-1})$
   (c) $|\mathring{\nabla}\beta_r|_{\mathring{H}^2} = O(r^{-1})$
4. $M = m_0 + \tilde{m}$ where $m_0 > 0$ is constant independent of δ and $|\tilde{m}|_{\mathring{H}^2} \leqslant \delta h_3(r)$ for $h_3 = o(1)$
5. η is a transversal 1-form satisfying:
   (a) $\eta = o_2(1)$
   (b) $|\eta|_{\mathring{H}^3} + r|\mathcal{L}_{\partial_r}\tilde{\eta}|_{\mathring{H}^3} \leqslant \delta h_4(r)$ for $h_4 = o(1)$.

6.2.1 A geodesic foliation

As in the spherically symmetric case we identify the null geodesic generator $Dv = e^{-\beta}\partial_r$. We will again for convenience take the background foliation to be level sets of the coordinate r. We wish therefore to relate the given decay in r to the geodesic foliation given by the generator $\underline{L} := Dv$ in order to show Ω is asymptotically flat with strong flux decay.

Once again $\frac{ds}{dr} = e^{\beta} = 1 + f$ where $f = \beta\frac{e^{\beta}-1}{\beta}$ is r-integrable due to decay condition 3. Taking local coordinates (ϑ, φ) on Σ_{r_0} (for some r_0) we have

$$s = r - c_0(\vartheta, \varphi) - \beta_1(r, \vartheta, \varphi) \tag{6.5}$$

for $\beta_0(\vartheta, \varphi) := \int_{r_0}^{\infty} f(t, \vartheta, \varphi)dt$, $c_0 = r_0 - \beta_0$ and $\beta_1(r, \vartheta, \varphi) = \int_r^{\infty} f(t, \vartheta, \varphi)dt$. Since each Σ_r is compact, an m-th order partial derivative of f is bounded by $C|\mathring{\nabla} f|_{\mathring{H}^{m-1}}$ for some constant C independent of r (from decay condition 3). From decay condition 3, provided $m \leqslant 4$, derivatives in ϑ, φ of β_0 and β_1 pass into the integral (for fixed r) onto f and are bounded. On any Σ_s (i.e fixed s) it follows from (6.5) that

$$\partial_{\vartheta(\varphi)} r = -\frac{\int_{r_0}^{r} \partial_{\vartheta(\varphi)} f(t, \vartheta, \varphi)dt}{1+f} = -e^{-\beta}\int_{r_0}^{r} \beta_{\vartheta(\varphi)} e^{\beta} dt$$

with bounded derivatives up to third order. It's a simple verification in local coordinates, from

$$r(s, \vartheta, \varphi) = s + c_0(\vartheta, \varphi) + \beta_1(r(s, \vartheta, \varphi), \vartheta, \varphi),$$

that $\partial_s^i \beta_1 = o_{3-i}^X(s^{-i})$ for $0 \leqslant i \leqslant 3$. Coupled with the fact that $\mathcal{L}_{\underline{L}} = e^{-\beta}\mathcal{L}_{\partial_r}$ on transversal tensors we conclude that $(\mathcal{L}_{\underline{L}})^i\tilde{\gamma} = o_{3-i}^X(s^{1-i})$ for $0 \leqslant i \leqslant 3$ and therefore

$$\gamma_s = r^2\gamma|_{\Sigma_s} = s^2\mathring{\gamma} + s\Gamma_1 + \tilde{\Gamma} \tag{6.6}$$

where

$$\Gamma_1 = 2c_0\mathring{\gamma} + \delta\gamma_1$$

$$\tilde{\Gamma} = \tilde{\gamma} + 2s\beta_1\mathring{\gamma} + c_0^2\mathring{\gamma} + c_0\delta\gamma_1 + \beta_1^2\mathring{\gamma} + 2c_0\beta_1\mathring{\gamma} + \beta_1\delta\gamma_1$$

satisfies the requirements towards strong decay.

6.2.2 Calculating ρ

Since we will compare computations for the foliation $\{\Sigma_r\}$ with the geodesic foliation of 5.4.1 we will revert back to denoting the covariant derivative on Σ_s by ∇ and the covariant derivative on Σ_r by $\not\nabla$. For the foliation $\{\Sigma_r\}$ we have the linearly independent normal vector fields

$$\underline{L} = e^{-\beta}\partial_r$$
$$Dr = e^{-\beta}\partial_v + (\mathfrak{h} + \alpha + |\vec{\eta}|^2)\partial_r - \vec{\eta}$$

from which similar calculations as in spherical symmetry yield the unique null normal satisfying $\langle \underline{L}, L\rangle = 2$ to be given by

$$L = 2\partial_v + e^{\beta}(\mathfrak{h} + \alpha + |\vec{\eta}|^2)\partial_r - 2e^{\beta}\vec{\eta}.$$

Lemma 6.2.1. *We have*

$$\underline{\chi} = e^{-\beta}(r\mathring{\gamma} + \frac{\delta}{2}\gamma_1 + \frac{1}{2}(\mathcal{L}_{\partial_r}\tilde{\gamma}))$$
$$\operatorname{tr}\underline{\chi} = e^{-\beta}(\frac{2}{r} + \frac{\delta\underline{\theta}}{r^2}) + \delta o_4^X(r^{-2})$$

for $\underline{\theta} := -\frac{1}{2}\mathring{\operatorname{tr}}\gamma_1$*. Moreover,*

$$\not\nabla^m\underline{\chi} = -e^{-\beta}\frac{\delta}{2}\mathring{\nabla}^m\gamma_1 + \delta o_{4-m}^X(1), \;\; 0 \leqslant m \leqslant 4.$$

Proof. First we extend $V, W \in E_{\partial_r}(\Sigma_{r_0})$ off of Ω such that $[\partial_v, V(W)] = 0$. Then for $\underline{\chi}$:

$$\underline{\chi}(V,W) = \langle D_V(e^{-\beta}\partial_r), W\rangle$$
$$= e^{-\beta}\langle D_V\partial_r, W\rangle$$

$$= e^{-\beta}\frac{1}{2}\partial_r\langle V, W\rangle$$

$$= e^{-\beta}(r\mathring{\gamma}(V,W) + \frac{\delta}{2}\gamma_1(V,W) + \frac{1}{2}\mathcal{L}_{\partial_r}\tilde{\gamma}(V,W))$$

having used the Koszul formula to get the third line. So using a basis extension $\{X_1, X_2\} \subset E_{\partial_r}(\Sigma_{r_0})$ Proposition 5.3.1 provides the inverse metric

$$\gamma(r)^{ij} = \frac{1}{r^2}\mathring{\gamma}^{ij} - \frac{\delta}{r^3}\mathring{\gamma_1}^{ij} + \delta o_5^X(r^{-3})$$

and $\operatorname{tr}\underline{\chi}$ follows by contracting $\gamma(r)^{-1}$ with $\underline{\chi}$. For the final identity we note from Lemma 5.3.2 we have for the decomposition $\gamma_r = r^2\mathring{\gamma} + r\delta\gamma_1 + \tilde{\gamma}$ the difference tensor

$$\begin{aligned}\langle\mathcal{D}(V,W), U\rangle &= \langle\not\nabla_V W - \mathring{\nabla}_V W, U\rangle \\ &= \frac{r\delta}{2}\Big(\mathring{\nabla}_V\gamma_1(W,U) + \mathring{\nabla}_W\gamma_1(V,U) - \mathring{\nabla}_U\gamma_1(V,W)\Big) \\ &\quad + \frac{1}{2}\Big(\mathring{\nabla}_V\tilde{\gamma}(W,U) + \mathring{\nabla}_W\tilde{\gamma}(V,U) - \mathring{\nabla}_U\tilde{\gamma}(V,W)\Big)\end{aligned}$$

for $V, W, U \in E(\Sigma_{r_0})$. So proceeding as in Proposition 5.3.5

$$\begin{aligned}\not\nabla_i\underline{\chi}_{jk} &= \mathring{\nabla}_i\underline{\chi}_{jk} - \mathcal{D}_{ij}^m\underline{\chi}_{mk} - \mathcal{D}_{ik}^m\underline{\chi}_{jm} \\ &= \mathring{\nabla}_i(re^{-\beta}\mathring{\gamma}_{jk} + e^{-\beta}\frac{\delta}{2}\gamma_{1jk} + e^{-\beta}\frac{1}{2}(\mathcal{L}_{\partial_r}\tilde{\gamma})_{jk}) - e^{-\beta}\delta\mathring{\nabla}_i(\gamma_{1jk}) + \delta o_4^X(1) \\ &= r\mathring{\gamma}_{jk}\mathring{\nabla}_i(e^{-\beta}) - e^{-\beta}\frac{\delta}{2}\mathring{\nabla}_i\gamma_{1jk} + \frac{\delta}{2}\gamma_{1jk}\mathring{\nabla}_i(e^{-\beta}) + \delta o_3^X(1) \\ &= -e^{-\beta}\frac{\delta}{2}\mathring{\nabla}_i\gamma_{1jk} + \delta o_3^X(1).\end{aligned}$$

Iteration provides our result

$$\not\nabla^m\underline{\chi} = -e^{-\beta}\frac{\delta}{2}\mathring{\nabla}^m\gamma_{1jk} + \delta o_{4-m}^X(1), \;\; 1 \leqslant m \leqslant 4$$

from decay condition 3. □

For χ we have

$$\chi(V,W) = 2\langle D_V\partial_v, W\rangle + e^{\beta}(\mathfrak{h}+\alpha+|\vec{\eta}|^2)\langle D_V\partial_r, W\rangle - 2\langle D_V e^{\beta}\vec{\eta}, W\rangle$$
$$= 2\langle D_V\partial_v, W\rangle + e^{2\beta}(\mathfrak{h}+\alpha+|\vec{\eta}|^2)\underline{\chi}(V,W) - 2\nabla\!\!\!/_V(e^{\beta}\eta)(W)$$

and using the Koszul formula on the first term we see

$$2\langle D_V\partial_v, W\rangle$$
$$= V(e^{\beta}\eta(W)) + \partial_v\langle V,W\rangle - W(e^{\beta}\eta(V)) - \langle V,[\partial_v,W]\rangle + \langle\partial_v,[W,V]\rangle + \langle W,[V,\partial_v]\rangle$$
$$= \nabla\!\!\!/_V(e^{\beta}\eta)(W) - \nabla\!\!\!/_W(e^{\beta}\eta)(V)$$
$$= \operatorname{curl}(e^{\beta}\eta)(V,W)$$

so that a trace over V,W yields $\operatorname{tr}\chi = e^{2\beta}(\mathfrak{h}+\alpha+|\vec{\eta}|^2)\operatorname{tr}\underline{\chi} - 2\nabla\!\!\!/\cdot(e^{\beta}\eta)$ and therefore

$$\langle\vec{H},\vec{H}\rangle = e^{2\beta}(\mathfrak{h}+\alpha+|\vec{\eta}|^2)(\operatorname{tr}\underline{\chi})^2 - 2\nabla\!\!\!/\cdot(e^{\beta}\eta)\operatorname{tr}\underline{\chi}$$
$$= \left(1-\frac{2M}{r}+\delta\frac{\alpha_0}{r}\right)\left(\frac{2}{r}+\frac{\delta\underline{\theta}}{r^2}\right)^2 + \delta o_2^X(r^{-3})$$
$$= \left(1-\frac{2M}{r}+\delta\frac{\alpha_0}{r}\right)\left(\frac{4}{r^2}+\frac{4\delta\underline{\theta}}{r^3}\right) + \delta o_2^X(r^{-3})$$
$$= \frac{4}{r^2}\left(1-\frac{2m_0}{r}+\delta\frac{\underline{\theta}}{r}+\delta\frac{\alpha_0}{r}\right) + \delta o_2^X(r^{-3})$$

from decay conditions 2-5. For ζ we have

$$\zeta(V) = \langle D_V(e^{-\beta}\partial_r),\partial_v\rangle - e^{\beta}\langle D_V(e^{-\beta}\partial_r),\vec{\eta}\rangle$$
$$= -V\beta + e^{-\beta}\langle D_V\partial_r,\partial_v\rangle - \langle D_V\partial_r,\vec{\eta}\rangle$$
$$= -V\beta + e^{-\beta}\langle D_V\partial_r,\partial_v\rangle - e^{\beta}\underline{\chi}(V,\vec{\eta}).$$

From the Koszul formula

$$2\langle D_V\partial_r,\partial_v\rangle$$
$$= V\langle\partial_r,\partial_v\rangle + \partial_r\langle V,\partial_v\rangle - \partial_v\langle V,\partial_r\rangle - \langle V,[\partial_r,\partial_v]\rangle + \langle\partial_r,[\partial_v,V]\rangle + \langle\partial_v,[V,\partial_r]\rangle$$
$$= e^{\beta}V\beta + \partial_r(e^{\beta}\eta(V))$$

$$= e^{\beta} V\beta + \mathcal{L}_{\partial_r}(e^{\beta}\eta)(V)$$

from which we conclude that $\zeta(V) = -\frac{1}{2}V(\beta) + \frac{e^{-\beta}}{2}\mathcal{L}_{\partial_r}(e^{\beta}\eta)(V) - e^{\beta}\underline{\chi}(V, \vec{\eta})$ and

$$\begin{aligned}
\nabla\!\!\!/ \cdot \zeta &= -\frac{1}{2}\Delta\!\!\!/\beta + \frac{1}{2}\nabla\!\!\!/ \cdot (e^{-\beta}\mathcal{L}_{\partial_r}(e^{\beta}\eta)) - \nabla\!\!\!/ \cdot (e^{\beta}\underline{\chi}(\vec{\eta})) \\
&= -\frac{1}{2}\Delta\!\!\!/\beta + \frac{1}{2}\nabla\!\!\!/ \cdot (\beta_r\eta) + \frac{1}{2}\nabla\!\!\!/ \cdot (\mathcal{L}_{\partial_r}\eta) - e^{\beta}\underline{\chi}(\nabla\!\!\!/\beta, \vec{\eta}) - e^{\beta}\nabla\!\!\!/ \cdot (\underline{\chi}(\vec{\eta})) \\
&= \delta o_2^X(r^{-3})
\end{aligned}$$

having used decay conditions 3, 5 and Lemma 6.2.1 for the final line.

Lemma 6.2.2. *Ω satisfies conditions 1, 2 and 3 of Definition 5.3.1. Ω additionally satisfies strong flux decay if and only if*

$$\frac{1}{2}\mathring{\nabla} \cdot \gamma_1 + d\underline{\theta} = 0$$

for $\underline{\theta} = -\frac{1}{2}\mathring{\mathrm{tr}}\gamma_1$ and is subsequently past asymptotically flat.

Proof. Having already verified condition 1 up to strong decay for γ_s of our geodesic foliation $\{\Sigma_s\}$ we continue to show conditions 2 and 3.
Given $V \in E_{\partial_r}(\Sigma_{r_0})$ Lemma 5.1.1 ensures $V - Vs\underline{L}|_{\Sigma_s} \in \Gamma(T\Sigma_s)$ and we see that

$$\begin{aligned}
[V - Vs\underline{L}, \underline{L}] &= [V, \underline{L}] + \underline{L}Vs\underline{L} \\
&= e^{\beta}V(e^{-\beta})\underline{L} + e^{-\beta}V(\partial_r s)\underline{L} \\
&= (e^{\beta}V(e^{-\beta}) + e^{-\beta}V(e^{\beta}))\underline{L} \\
&= 0.
\end{aligned}$$

So $V - Vs\underline{L} \in E(\Sigma_0)$ and Lemma 5.1.1 gives

$$\begin{aligned}
t(V - Vs\underline{L}) = t(V) &= \zeta(V) + \underline{\chi}(V, \nabla\!\!\!/ s) \\
&= -\frac{1}{2}V(\beta) + \frac{1}{2}\beta_r\eta(V) + \frac{1}{2}(\mathcal{L}_{\partial_r}\eta)(V) - e^{\beta}\underline{\chi}(V, \vec{\eta}) + \underline{\chi}(V, \nabla\!\!\!/ s) \\
&= (\delta o_3^X(r^{-1}) \cap o_1(r^{-1}))(V) + \underline{\chi}(V, \nabla\!\!\!/ s)
\end{aligned}$$

$$= (\delta o_3^X(r^{-1}) \cap o_1(r^{-1}))(V) + re^{-\beta}\mathring{\gamma}(V, \frac{1}{r^2}\mathring{\nabla} s)$$

$$= \frac{e^{-\beta}}{r} V\beta_0 + (\delta o_3^X(r^{-1}) \cap o_1(r^{-1}))(V)$$

having used decay conditions 3 and 5 to get the second line, Lemma 6.2.1 for the third and (6.5) for the last. Moreover,

$$\begin{aligned}
(\mathcal{L}_{V-Vs\underline{L}} t)(W - Ws\underline{L}) &= (V - Vs\underline{L})(t(W - Ws\underline{L})) - t([V, W]) \\
&= (\mathcal{L}_V t)(W) - Vs\underline{L}(t(W)) \\
&= (\mathcal{L}_V - e^{-\beta} V s \mathcal{L}_{\partial_r})(\frac{d\beta_0}{r})(W) + o(r^{-1}) \\
&= \frac{1}{r}\mathcal{L}_V(d\beta_0)(W) + o(r^{-1}) \\
&= \frac{1}{r}(\mathcal{L}_{V-Vs\underline{L}} d\beta_0)(W - Ws\underline{L}) + o(r^{-1})
\end{aligned}$$

where the last line follows since β_0 is $\underline{L}$-Lie constant. With a basis extension $\{X_i\} \subset E(\Sigma_0)$ we therefore conclude that $\mathcal{L}_{X_i} t = \frac{1}{s}\mathcal{L}_{X_i} d\beta_0 + o(s^{-1})$ so that condition 2 for asymptotic flatness is satisfied up to strong decay with $t_1 = d\beta_0$. From Proposition 5.3.1 and (6.6):

$$\begin{aligned}
\operatorname{tr} K &= \frac{2}{s} - \frac{1}{2s^2}\mathring{\operatorname{tr}}\Gamma_1 + o(s^{-2}) \\
&= \frac{2}{s} - \frac{1}{2s^2}\mathring{\operatorname{tr}}(2c_0\mathring{\gamma} + \delta\gamma_1) + o(s^{-2}) \\
&= \frac{2}{s} + \frac{\delta\underline{\theta} - 2c_0}{s^2} + o(s^{-2})
\end{aligned}$$

and

$$\begin{aligned}
\hat{K} &= K - \frac{1}{2}\operatorname{tr} K\gamma_s \\
&= s\mathring{\gamma} + \frac{1}{2}\Gamma_1 - \frac{1}{2}\Big(\frac{2}{s} + \frac{\delta\underline{\theta} - 2c_0}{s^2} + o(s^{-2})\Big)\gamma_s + o(1) \\
&= s\mathring{\gamma} + \frac{1}{2}(2c_0\mathring{\gamma} + \delta\gamma_1) - \frac{1}{2}(\frac{2}{s} + \frac{\delta\underline{\theta} - 2c_0}{s^2})(s^2\mathring{\gamma} + s(2c_0\mathring{\gamma} + \delta\gamma_1)) + o(1)
\end{aligned}$$

$$= -\frac{\delta}{2}(\gamma_1 + \underline{\theta}\mathring{\gamma}) + o(1).$$

For condition 3 we take $r|_{\Sigma_s} \in \mathcal{F}(\Sigma_s)$ and Lie drag it to the the rest of Ω along ∂_r (hence $\underline{L}$) to give $r_s \in \mathcal{F}(\Omega)$. Using Lemma 5.1.1 from the vantage point of the cross-section Σ_s amongst the background foliation $\{\Sigma_r\}$:

$$e^{-\beta}\operatorname{tr} Q = e^{-\beta}\operatorname{tr}\chi - 4(\zeta + \not{d}\log e^{\beta})(\not{\nabla} r_s) - 2\Delta r_s + |\not{\nabla} r_s|^2 e^{\beta}\operatorname{tr}\underline{\chi} - 2\beta_r|\not{\nabla} r_s|^2$$

From the expression of $r(s)$ in 5.4.1, recalling Remark 5.3.2, we see $dr_s = -d\beta_0 + o(1)$ from which Lemma 5.3.2 implies that $\Delta r_s = -\frac{1}{s^2}\mathring{\Delta}\beta_0 + o(s^{-2})$. From decay conditions 3, 5 and Lemma 6.2.1 we have

$$\begin{aligned}
\operatorname{tr} Q &= \operatorname{tr}\chi|_{\Sigma_s} + 2\frac{\mathring{\Delta}\beta_0}{s^2} + o(s^{-2}) \\
&= \Big(e^{2\beta}(\mathfrak{h} + \alpha + |\vec{\eta}|^2)\operatorname{tr}\underline{\chi} - 2\not{\nabla}\cdot(e^{\beta}\eta)\Big)|_{\Sigma_s} + 2\frac{\mathring{\Delta}\beta_0}{s^2} + o(s^{-2}) \\
&= (\frac{2}{s} + \frac{\delta\underline{\theta} - 2c_0}{s^2})(1 - \frac{2M}{s} + \delta\frac{\alpha_0}{s}) + 2\frac{\mathring{\Delta}\beta_0}{s^2} + o(s^{-2}) \\
&= \frac{2}{s} + \frac{\delta\underline{\theta} - 2c_0 - 4M + 2\delta\alpha_0}{s^2} + 2\frac{\mathring{\Delta}\beta_0}{s^2} + o(s^{-2}) \\
&= \frac{2}{s} - 2\frac{c_0 + 2M}{s^2} + \frac{1}{s^2}(\delta\underline{\theta} + 2\mathring{\Delta}\beta_0 + 2\delta\alpha_0) + o(s^{-2})
\end{aligned}$$

and condition 3 follows as soon as we set $M = m_0 + \delta o_2^X(1)$. As in the spherically symmetric case the highest order term for $\operatorname{tr} Q$ agrees with $\frac{2\mathring{\mathcal{K}}}{s}$ where $\mathring{\mathcal{K}} = 1$ is the Gaussian curvature of $\mathring{\gamma}$. We recall that our use of condition 4 depends on whether Ω has strong flux decay (Remark 5.3.5). From Proposition 5.3.1 and (6.6) we will have strong flux decay if and only if

$$\begin{aligned}
d\beta_0 = t_1 &= \frac{1}{2}\mathring{\nabla}\cdot\Gamma_1 - \frac{1}{2}\not{d}\mathring{\operatorname{tr}}\Gamma_1 \\
&= \frac{1}{2}\mathring{\nabla}\cdot(2c_0\mathring{\gamma} + \delta\gamma_1) + \not{d}(\delta\underline{\theta} - 2c_0)
\end{aligned}$$

$$= \frac{1}{2}d(-2\beta_0) + \frac{\delta}{2}\mathring{\nabla}\cdot\gamma_1 + \delta d\underline{\theta} + 2d\beta_0$$

$$= d\beta_0 + \delta(\frac{1}{2}\mathring{\nabla}\cdot\gamma_1 + d\underline{\theta})$$

which in turn holds if and only if $\frac{1}{2}\mathring{\nabla}\cdot\gamma_1 + d\underline{\theta} = 0$. □

Henceforth we will adopt the conditions of Lemma 6.2.2 for Ω. From Proposition 5.3.2

$$\begin{aligned}\mathcal{K}_{r^2\gamma} &= \frac{1}{r^2} + \frac{\delta}{r^3}\Big(\underline{\theta} + \frac{1}{2}\mathring{\nabla}\cdot\mathring{\nabla}\cdot\gamma_1 + \mathring{\Delta}\underline{\theta}\Big) + \delta o_4^X(r^{-3}) \\ &= \frac{1}{r^2} + \frac{\delta}{r^3}\underline{\theta} + \delta o_4^X(r^{-3}).\end{aligned}$$

From Lemma 6.2.1 we have

$$\not{\nabla}_i\not{\nabla}_j\underline{\chi}_{mn} = -\frac{\delta}{2}\mathring{\nabla}_i\mathring{\nabla}_j\gamma_{1mn} + \delta o_2^X(1)$$

so that contraction with $\gamma(r)^{-1}$ first in mn then ij gives

$$\not{\Delta}\operatorname{tr}\underline{\chi} = \frac{\delta}{r^4}\mathring{\Delta}\underline{\theta} + \delta o_2^X(r^{-4})$$

which we use in $\not{\Delta}\log\operatorname{tr}\underline{\chi} = \frac{\not{\Delta}\operatorname{tr}\underline{\chi}}{\operatorname{tr}\underline{\chi}} - \frac{|\not{\nabla}\operatorname{tr}\underline{\chi}|^2}{(\operatorname{tr}\underline{\chi})^2}$ to conclude

$$\not{\Delta}\log\operatorname{tr}\underline{\chi} = \frac{\delta}{2r^3}\mathring{\Delta}\underline{\theta} + \delta o_2^X(r^{-3}).$$

Finally we have ρ

$$\begin{aligned}\rho &= \mathcal{K}_{r^2\gamma} - \frac{1}{4}\langle\vec{H},\vec{H}\rangle + \not{\nabla}\cdot\zeta - \not{\Delta}\log\operatorname{tr}\underline{\chi} \\ &= \frac{1}{r^2} + \frac{\delta}{r^3}\underline{\theta} - \frac{1}{r^2} + \frac{2m_0}{r^3} - \delta\frac{\underline{\theta}}{r^3} - \delta\frac{\alpha_0}{r^3} - \frac{\delta}{2r^3}\mathring{\Delta}\underline{\theta} + \delta o_2^X(r^{-3}) \\ &= \frac{2m_0}{r^3} - \frac{\delta}{r^3}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0) + \delta o_2^X(r^{-3})\end{aligned}$$

$$= \frac{2m_0}{r^3} - \frac{\delta}{r^3}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0) + \delta o_2^X(r^{-3})$$

and

$$\frac{1}{4}\langle \vec{H}, \vec{H}\rangle - \frac{1}{3}\not\Delta \log\rho = \frac{1}{r^2}\Big(1 - \frac{2m_0}{r} + \delta\frac{\underline{\theta}}{r} + \delta\frac{\alpha_0}{r}\Big)$$
$$- \frac{1}{3}\not\Delta \log\Big(\frac{2m_0}{r^3} - \frac{\delta}{r^3}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0) + \delta o_2^X(r^{-3})\Big) + \delta o_2^X(r^{-3}).$$

We may now use Lemma 5.3.2 to decompose the last term

$$\not\Delta \log\Big(\frac{2m_0}{r^3} - \frac{\delta}{r^3}(\frac{5}{2}\mathring{\Delta}\underline{\theta} + \alpha_0) + \delta o_2^X(r^{-3})\Big)$$
$$= \frac{1}{r^2}\mathring{\Delta}\log\Big(1 - \frac{\delta}{2m_0}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0) + \delta o_2^X(1)\Big) + \delta o(r^{-2})$$
$$= \frac{1}{r^2}\mathring{\Delta}\log\Big(1 - \frac{\delta}{2m_0}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0)\Big) + \delta o(r^{-2})$$

giving

$$\frac{1}{4}\langle \vec{H}, \vec{H}\rangle - \frac{1}{3}\not\Delta \log\rho = \frac{1}{r^2}\Big(1 - \frac{2m_0}{r} - \frac{1}{3}\mathring{\Delta}\log\Big(1 - \frac{\delta}{2m_0}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0)\Big)\Big) + \delta o(r^{-2}).$$

Since $m_0 > 0$ we notice for sufficiently small δ our perturbation ensures $\rho > 0$ for all $r > 0$. However, from our construction so far it's not yet possible to conclude that some $\delta > 0$ will enforce $\frac{1}{4}\langle \vec{H}, \vec{H}\rangle \geqslant \frac{1}{3}\not\Delta \log\rho$ along the foliation. Moreover, the existence of a horizon ($\operatorname{tr}\chi = 0$) is equally questionable.

6.2.3 Smoothing to Spherical Symmetry

We will solve this difficulty by 'smoothing' away all perturbations in a neighborhood of the (desired) horizon in order to obtain spherical symmetry on $r < r_1$ for some $r_1 > 0$ yet to be chosen. The resulting spherical symmetry will uncover the horizon at $r = r_0 < r_1$ and will also provide a choice of $\delta > 0$ so that $\frac{1}{4}\langle \vec{H}, \vec{H}\rangle > \frac{1}{3}\not\Delta \log\rho$ away from it, causing the foliation $\{\Sigma_r\}$ to be an (SP)-foliation.

We will use a smooth step function $0 \leqslant S_\delta(r) \leqslant 1$ such that $S_\delta(r) = 0$ for $r < r_1$ and $S_\delta(r) = 1$ for $r > r_2$ for some finite $r_2(\delta)$ chosen to ensure $|S'_\delta(r)| \leqslant \delta$. By first choosing *parameter functions* for the desired spherically symmetric region; $\tilde{\beta}(v, r)$ and $0 < \tilde{M}(v, r) = m_0 + o(1)$ such that $r_0 = 2\tilde{M}(v_0, r_0)$ and $2\tilde{M}(v_0, r) < r$ for $r > r_0$ we induce spherical symmetry on $r < r_1$ with the following substitutions:

$$\tilde{\gamma} \to \delta r(S_\delta(r) - 1)\gamma_1 + S_\delta(r)\tilde{\gamma}$$

$$\beta(r, \vartheta, \varphi) \to S_\delta(r)\beta(r, \vartheta, \varphi) + (1 - S_\delta(r))\tilde{\beta}(v_0, r)$$

$$M(r, \vartheta, \varphi) \to S_\delta(r)M(r, \vartheta, \varphi) + (1 - S_\delta(r))\tilde{M}(v_0, r)$$

$$\tilde{\alpha} \to S_\delta(r)\tilde{\alpha} - (1 - S_\delta(r))\frac{\delta\alpha_0}{r}$$

$$\eta \to S_\delta(r)\eta.$$

We leave the reader the simple verification that these changes to our perturbation tensors $\tilde{\gamma}$, β, M, $\tilde{\alpha}$ and η maintain the decay conditions 1-5. Clearly for $r > r_2$ our substitutions leave the metric unchanged while inducing spherical symmetry on $r < r_1$ with the spherical parameter functions $\tilde{\beta}\,, \tilde{M}$:

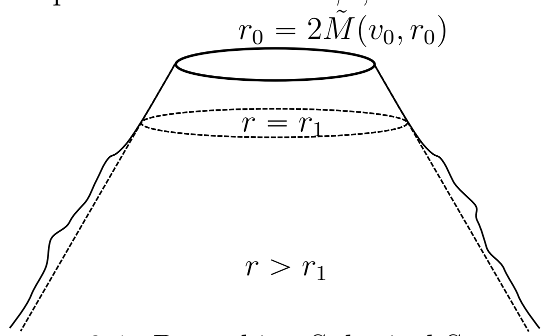

FIGURE 6.1: Perturbing Spherical Symmetry

An example $S_\delta(r)$ is given by the function

$$S_\delta(r) = \begin{cases} 0 & r \leqslant r_1 \\ \dfrac{e^{\frac{k}{r_1 - r}}}{e^{\frac{k}{r_1 - r}} + e^{\frac{k}{r - r_2}}} & r_1 < r < r_2 \\ 1 & r_2 \leqslant r \end{cases}$$

where $k = \frac{4e^4}{\delta}$ and $r_2(\delta) = r_1 + k$. Since $S_\delta(r) = P(\frac{1}{r_1 - r} + \frac{1}{r_2 - r})$ for $P(r) = \frac{e^{kr}}{1+e^{kr}}$ satisfying the logistic equation

$$P'(r) = kP(1-P)$$

we have

$$\begin{aligned} S'_\delta(r) &= kS_\delta(r)(1 - S_\delta(r))(\frac{1}{(r-r_1)^2} + \frac{1}{(r-r_2)^2}) \\ &= k\frac{S_\delta(r)}{(r-r_1)^2}(1 - S_\delta(r)) + kS_\delta(r)\frac{1-S_\delta(r)}{(r-r_2)^2} \\ &\leqslant k\Big(\frac{S_\delta(r)}{(r-r_1)^2} + \frac{1-S_\delta(r)}{(r-r_2)^2}\Big). \end{aligned}$$

Elementary analysis reveals on the interval $r_1 < r < r_2$ that

$$0 \leqslant \frac{e^{\frac{k}{r_1 - r}}}{(r-r_1)^2} \leqslant \frac{4e^2}{k^2}$$

$$0 \leqslant \frac{1}{e^{\frac{k}{r_1-r}} + e^{\frac{k}{r-r_2}}} \leqslant \frac{1}{2}e^2$$

yielding from simple symmetry arguments that both $\frac{S_\delta(r)}{(r-r_1)^2}, \frac{1-S_\delta(r)}{(r-r_2)^2} \leqslant \frac{2e^4}{k^2}$ and therefore

$$0 \leqslant S'_\delta(r) \leqslant k\frac{4e^4}{k^2} = \delta$$

as desired. Denoting $m(r,\delta) := S_\delta(r)m_0 + (1 - S_\delta(r))\tilde{M}(r)$ the new metric gives

$$\rho = \begin{cases} \frac{2\tilde{M}(v_0,r)}{r^3}, & r < r_1 \\ \frac{2m(r,\delta)}{r^3} - \frac{\delta}{r^3}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0) + \delta o_2^X(r^{-3}), & r_1 \leqslant r \leqslant r_2 \\ \frac{2m_0}{r^3} - \frac{\delta}{r^3}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0) + \delta o_2^X(r^{-3}), & r_2 < r \end{cases}$$

and $\frac{1}{4}\langle\vec{H},\vec{H}\rangle - \frac{1}{3}\Delta\!\!\!/ \log\rho =$

$$\begin{cases} \frac{1}{r^2}(1 - \frac{2\tilde{M}(v_0,r)}{r}), & r < r_1 \\ \frac{1}{r^2}\Big(1 - \frac{2m(r,\delta)}{r} - \frac{1}{3}\mathring{\Delta}\log\Big(1 - \frac{\delta}{2m(r,\delta)}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0)\Big)\Big) + \delta o(r^{-2}), & r_1 \leqslant r \leqslant r_2 \\ \frac{1}{r^2}\Big(1 - \frac{2m_0}{r} - \frac{1}{3}\mathring{\Delta}\log\Big(1 - \frac{\delta}{2m_0}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0)\Big)\Big) + \delta o(r^{-2}), & r_2 < r. \end{cases}$$

Since $C(r_1) \geqslant m(r,\delta) \geqslant m_0$ for $C(r_1) := \sup_{r_0<r<r_1} \tilde{M}$ we see for any choice of $r_1 > C(r_1)$ (which is possible since $\tilde{M} = m_0 + o(1)$) and sufficiently small δ the foliation $\{\Sigma_r\}$ satisfies property (SP). If we therefore restrict to perturbations satisfying the dominant energy condition on Ω then Theorem 2.1.2 implies the following:

Theorem 6.2.1. *Let g_δ be a metric perturbation off of spherical symmetry given by*

$$g_\delta = -(\mathfrak{h} + \alpha)e^{2\beta} dv \otimes dv + e^{\beta}(dv \otimes (dr + \eta) + (dr + \eta) \otimes dv) + r^2\gamma$$

where

1. $r^2\gamma = r^2\mathring{\gamma} + r\delta\gamma_1 + \tilde{\gamma}$ *is trasversal with* $\mathring{\gamma}$ *the transversal* ∂_r*-Lie constant round metric on* $\mathbb{S}^2$ *independent of* δ*,* γ_1 *a transversal* ∂_r*-Lie constant 2-tensor independent of* δ *satisfying* $\mathring{\nabla} \cdot \gamma_1 = d(\mathring{\mathrm{tr}}\gamma_1)$ *and* $(\mathcal{L}_{\partial_r})^i\tilde{\gamma} = \delta o^X_{5-i}(r^{1-i})$ *for* $0 \leqslant i \leqslant 3$.

2. $\alpha = \delta\frac{\alpha_0}{r} + \tilde{\alpha}$ *where* α_0 *is* ∂_r*-constant, independent of* δ *and* $|\tilde{\alpha}|_{\mathring{H}^2} \leqslant \delta h_1(r)$ *for* $h_1 = o(r^{-1})$

3. β *satisfies:*

   (a) $|\beta| = o_2(r^{-1})$ *is r-integrable*

   (b) $|\mathring{\nabla}\beta|_{\mathring{H}^3} \leqslant \delta h_2(r)$ *for some integrable* $h_2 = o(r^{-1})$

   (c) $|\mathring{\nabla}\beta_r|_{\mathring{H}^2} = O(r^{-1})$

4. $M = m_0 + \tilde{m}$ *where* $m_0 > 0$ *is constant, independent of* δ *and* $|\tilde{m}|_{\mathring{H}^2} \leqslant \delta h_3(r)$ *for* $h_3 = o(1)$

5. η *is a transversal 1-form satisfying:*

   (a) $\eta = o_2(1)$

   (b) $|\eta|_{\mathring{H}^3} + r|\mathcal{L}_{\partial_r}\tilde{\eta}|_{\mathring{H}^3} \leqslant \delta h_4(r)$ *for* $h_4 = o(1)$.

Then for sufficiently small δ, $\Omega := \{v = v_0\}$ *is past asymptotically flat with strong flux decay. In addition, for any choice of spherical parameters* $\tilde{\beta}(v,r)$ *and* $\tilde{M}(v,r)$ *such that* $0 < \tilde{M}(v_0,r) = m_0 + o(1)$, $r_0 = 2\tilde{M}(v_0,r_0)$ *and* $2\tilde{M}(v_0,r) < r$ *for* $r > r_0$, *smoothing to spherical symmetry with the step function* $S_\delta(r)$ *(as above) according to:*

$$\tilde{\gamma} \to \delta r(S_\delta(r) - 1)\gamma_1 + S_\delta(r)\tilde{\gamma}$$

$$\beta(r,\vartheta,\varphi) \to S_\delta(r)\beta(r,\vartheta,\varphi) + (1 - S_\delta(r))\tilde{\beta}(r)$$

$$M(r,\vartheta,\varphi) \to S_\delta(r)M(r,\vartheta,\varphi) + (1 - S_\delta(r))\tilde{M}(r)$$

$$\tilde{\alpha} \to S_\delta(r)\tilde{\alpha} - (1 - S_\delta(r))\frac{\delta\alpha_0}{r}$$

$$\eta \to S_\delta(r)\eta$$

we have that $\Sigma := \{r_0 = 2\tilde{M}(v_0,r_0)\}$ *is marginally outer trapped and the coordinate spheres* $\{\Sigma_r\}_{r\geqslant r_0}$ *form an (SP)-foliation. Moreover, if* g_δ *respects the dominant energy condition on* Ω *we have the Penrose inequality:*

$$\sqrt{\frac{|\Sigma|}{16\pi}} \leqslant m_B$$

where m_B *is the Bondi mass of* Ω.

Appendix A

Bartnik data of coordinate spheres in boosted Schwarzschild

The Schwarzschild metric in isotropic coordinates is given by,

$$ds^2 = -\Big(\frac{\alpha}{\beta}\Big)^2 dt^2 + \beta^4(dx^2 + dy^2 + dz^2)$$

where $\alpha = 1 - \frac{M}{2R}$ and $\beta = 1 + \frac{M}{2R}$ for $R = \sqrt{x^2 + y^2 + z^2}$.

Alternatively making a change to spherical coordinates,

$$x = R \sin\vartheta \cos\varphi$$

$$y = R \sin\vartheta \sin\varphi$$

$$z = R \cos\vartheta$$

transforms the metric to

$$ds^2 = -\Big(\frac{\alpha}{\beta}\Big)^2 dt^2 + \beta^4\Big(dR^2 + R^2(d\vartheta^2 + (\sin\vartheta)^2 d\varphi^2)\Big).$$

Definition A.0.1. *For semi-Riemannian manifolds (B, g_B) and (F, g_F), given $f > 0$ a smooth function on B the warped product $\mathcal{M} = B \times_f F$ is the product manifold*

$B \times F$ *furnished with the metric tensor,*

$$g = \pi_1^\star(g_B) + (f \circ \pi_1)^2 \pi_2^\star(g_F)$$

where π_1 *and* π_2 *represent the canonical projections onto* B *and* F *respectively.*

For Warped Products we have a pointwise decomposition of each tangent space into orthogonal compliments $T_p\mathcal{M} = \ker(d(\pi_2)_p) \oplus \ker(d(\pi_1)_p)$ known as *normal* and *tangent* vectors (i.e. $T_p\mathcal{M} = \mathrm{nor}(T_p\mathcal{M}) \oplus \tan(T_p\mathcal{M})$) respectively. We also in the case of product manifolds recall the submodules $L(B), L(F) \subset \Gamma(T\mathcal{M})$ of *lifted* vector fields, whereby $X \in L(B) \subset \ker(d\pi_2)$ is the canonical representative of some $\tilde{X} \in \Gamma(TB)$ such that $d\pi_1(X) = \tilde{X}$. For every $(p, q) \in B \times_f F$ we refer to the semi-Riemanian submanifold $B \times q$ as a *leaf* and $p \times F$ as a *fibre.*
Therefore, returning to Schwarzschild geometry, we recognize a Warped Product structure $\mathbb{P} \times_{R\beta^2} \mathbb{S}^2$ with leaves isometric to $\mathbb{P} = \Big(\mathbb{R} \times (\frac{M}{2}, \infty), \big(\frac{\alpha}{\beta}\big)^2 dt^2 + \beta^4 dR^2\Big)$ and fibres homothetic to the standard round sphere via the function $R\beta^2$ on $\mathbb{P}$.

Introducing the boosted coordinates,

$$\begin{pmatrix} \cosh\psi & -\sinh\psi \\ -\sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} t \\ z \end{pmatrix} = \begin{pmatrix} \bar{t} \\ \bar{z} \end{pmatrix}$$

it is easily shown that the standard *boosted slice* $\{\bar{t} = 0\}$ (or $\{t - \tanh(\psi)z = 0\}$) inherits the induced metric:

$$ds^2 = \beta^4\Big(d\bar{x}^2 + d\bar{y}^2 + (\cosh^2\psi - \Big(\frac{\alpha}{\beta^3}\Big)^2 \sinh^2\psi) d\bar{z}^2\Big)$$

for $\bar{r} := \sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2} = R\sqrt{(1 - \tanh^2\psi\cos^2\vartheta)}$. It is also easy to show that the boosted time slice is asymptotically flat with the metric clearly displaying the required asymptotic behaviour for large $\bar{r}$.

Using the ambient Warped Product structure our goal is to study the geometry of the coordinate spheres, $\mathbb{S}^2_{\bar{r}}$, given by constant $\bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2}$ that foliates this boosted slice. We start by realizing $\mathbb{S}^2_{\bar{r}}$ as the intersection of the slices:

$$t = \tanh\psi R\cos\vartheta \tag{A.2a}$$

$$R^2 - t^2 = \bar{r}^2 \tag{A.2b}$$

yielding the equations

$$t(\vartheta) = \frac{\bar{r}\tanh\psi\cos\vartheta}{\sqrt{1-\tanh^2\psi\cos^2\vartheta}} \tag{A.3a}$$

$$R(\vartheta) = \frac{\bar{r}}{\sqrt{1-\tanh^2\psi\cos^2\vartheta}} \tag{A.3b}$$

or equivalently, for $\tanh\eta := \tanh\psi\cos\vartheta = \frac{t}{R}$,

$$t = \bar{r}\sinh\eta \tag{A.4a}$$

$$R = \bar{r}\cosh\eta \tag{A.4b}$$

(for constant $\bar{r}$).

From these equations we recognize a family of embeddings, $j_{\bar{r}} : \mathbb{S}^2 \to \mathbb{S}^2_{\bar{r}}$:

$$\begin{array}{ccc} \mathbb{S}^2 & \xrightarrow{f_{\bar{r}}} & \mathbb{P}\times_{R\beta^2}\mathbb{S}^2 \\ & \searrow^{id} & \downarrow \sigma \\ & & \mathbb{S}^2 \end{array}$$

$$(\vartheta,\varphi) \xrightarrow{f_{\bar{r}}} (t(\vartheta), R(\vartheta), \vartheta, \varphi) \xrightarrow{\sigma} (\vartheta,\varphi)$$

As such, given any $V \in \eta(T\mathbb{S}^2)$, by denoting $dj_{\bar{r}}(V)$ as $\bar{V}$, we have (assuming restriction to $\mathbb{S}^2_{\bar{r}}$ throughout):

$$\begin{aligned} \bar{V} &= V(t)\partial_t + V(R)\partial_R + V_{\mathbb{S}^2} \\ &= V\eta X + V_{\mathbb{S}^2} \end{aligned}$$

for $X = R\partial_t + t\partial_R$ and $V_{\mathbb{S}^2} \in L(\mathbb{S}^2)$.

Henceforth we will drop the subscript $V_{\mathbb{S}^2}$ and will refer interchangably between elements of $L(\mathbb{S}^2)$ and $\Gamma(T\mathbb{S}^2)$ as the meaning should remain clear from the context. Our use of the Warped Product structure will be extensively due to, ([21],pg.206)

Proposition A.0.1. *On a Warped Product $M = B \times_f F$, if $X, Y \in L(B)$ and $V, W \in L(F)$ then*

1. $D_X Y \in L(B)$ *is the lift of* $D_X Y$ *on* B.
2. $D_X V = D_V X = \frac{Xf}{f} V$
3. $nor D_V W = -\frac{\langle V,W \rangle}{f} Df$
4. $tan D_V W \in L(F)$ *is the lift of* $\nabla_V W$ *on* F.

Armed with this Proposition we extract some necessary results from the ambient spacetime

Corollary A.0.1.1. *For $X = R\partial_t + t\partial_R \in L(\mathbb{P})$ and $V, W \in L(\mathbb{S}^2)$,*

1. (a) $D_{\partial_R}\partial_R = -\dfrac{M}{R^2}\dfrac{1}{\beta}\partial_R$

   (b) $D_{\partial_R}\partial_t = D_{\partial_t}\partial_R = \dfrac{M}{R^2}\dfrac{1}{\alpha\beta}\partial_t$

   (c) $D_{\partial_t}\partial_t = (\dfrac{\alpha}{\beta^3})\dfrac{M}{R^2}\dfrac{1}{\beta^4}\partial_R$

2. $D_X X = (R + \dfrac{M}{\beta^4}\dfrac{\alpha}{\beta^3} - \dfrac{M}{\beta}(\dfrac{t}{R})^2)\partial_R + (t + \dfrac{2M}{\alpha\beta}\dfrac{t}{R})\partial_t$

3. $D_V W = \nabla_V W - \dfrac{\langle V, W\rangle}{R\beta^2} D(R\beta^2) = \nabla_V W - R\dfrac{\alpha}{\beta}(V, W)$

Where $\langle\cdot,\cdot\rangle$ denotes the ambient metric tensor, $(\cdot,\cdot)$ and ∇ the standard round metric and corresponding covariant derivative on the sphere $\mathbb{S}^2$.

Proof. We recall that, $\langle\partial_R, \partial_R\rangle = \beta^4$ and $\langle\partial_t, \partial_t\rangle = -(\frac{\alpha}{\beta})^2$:

1. • $\langle D_{\partial_R}\partial_R, \partial_R\rangle = \frac{1}{2}\partial_R\langle\partial_R, \partial_R\rangle = 2\beta^3(-\frac{M}{2R^2}) = -\frac{M\beta^3}{R^2}$

- $\langle D_{\partial_R}\partial_R, \partial_t\rangle = -\langle \partial_R, D_{\partial_R}\partial_t\rangle = -\frac{1}{2}\partial_t\langle \partial_R, \partial_R\rangle = 0$
- $\langle D_{\partial_R}\partial_t, \partial_R\rangle = \frac{1}{2}\partial_t\langle \partial_R, \partial_R\rangle = 0$
- $\langle D_{\partial_R}\partial_t, \partial_t\rangle = \frac{1}{2}\partial_R(-(\frac{\alpha}{\beta})^2) = -\frac{\alpha}{\beta^3}\frac{M}{R^2}$
- $\langle D_{\partial_t}\partial_t, \partial_R\rangle = -\frac{1}{2}\partial_R\langle \partial_t, \partial_t\rangle = \frac{\alpha}{\beta^3}\frac{M}{R^2}$
- $\langle D_{\partial_t}\partial_t, \partial_t\rangle = \frac{1}{2}\partial_t\langle \partial_t, \partial_t\rangle = 0$

Giving $D_{\partial_R}\partial_R = \frac{\langle D_{\partial_R}\partial_R, \partial_R\rangle}{\langle \partial_R, \partial_R\rangle}\partial_R + \frac{\langle D_{\partial_R}\partial_R, \partial_t\rangle}{\langle \partial_t, \partial_t\rangle}\partial_t = -\frac{M}{R^2\beta}\partial_R$, b) and c) follow similarly.

2. Since $D_X X = t^2 D_{\partial_R}\partial_R + t\partial_t + 2tRD_{\partial_R}\partial_t + R\partial_R + R^2 D_{\partial_t}\partial_t$ the result follows from 1.

3. Given $D_V W = \tan D_V W + \mathrm{nor} D_V W$, this is a simple application of Proposition A.0.1. and the fact that

$$\langle V, W\rangle = R^2\beta^4(d\pi_2 V, d\pi_2 W)\circ\pi_2$$

□

With all these tools in place we finally direct our attention towards $\mathbb{S}^2_{\bar{r}}$.

A.1 The Fundamental Forms of $\mathbb{S}^2_{\bar{r}}$

Proposition A.1.1.

$$D_{\bar{V}}\bar{W} = [V\eta W\eta\{R + \frac{M}{\beta^4}\frac{\alpha}{\beta^3} - \frac{M}{\beta}(\frac{t}{R})^2\} + tVW\eta - R\frac{\alpha}{\beta}(V,W)]\partial_R$$

$$+ [V\eta W\eta\{t + \frac{2M}{\alpha\beta}\frac{t}{R}\} + RVW\eta]\partial_t$$

$$+ \nabla_V W + \frac{\alpha}{\beta}\frac{t}{R}\{V\eta W + W\eta V\}$$

Proof.

$$\begin{aligned}
D_{\bar{V}}\bar{W} &= D_{V\eta X+V}(W\eta X + W) \\
&= V\eta W\eta D_X X + V\eta D_X W + VW\eta X + W\eta D_V X + D_V W \\
&= V\eta W\eta D_X X + \frac{X(R\beta^2)}{R\beta^2}(V\eta W + W\eta V) \\
&\quad + VW\eta X - \frac{\langle V, W\rangle}{R\beta^2} D(R\beta^2) + \nabla_V W
\end{aligned}$$

the final equality following from Proposition A.0.1. Now using Corollary A.0.1.1 we have,

$$\begin{aligned}
&= V\eta W\eta\{(R + \frac{M}{\beta^4}\frac{\alpha}{\beta^3} - \frac{M}{\beta}(\frac{t}{R})^2)\partial_R + (t + \frac{2M}{\alpha\beta}\frac{t}{R})\partial_t\} \\
&+ \frac{t\partial_R(R\beta^2)}{R\beta^2}(V\eta W + W\eta V) + VW\eta\{t\partial_R + R\partial_t\} - \frac{\langle V, W\rangle}{R\beta^2} D(R\beta^2) + \nabla_V W \\
&= V\eta W\eta\{(R + \frac{M}{\beta^4}\frac{\alpha}{\beta^3} - \frac{M}{\beta}\frac{t}{R})\partial_R + (t + \frac{2M}{\alpha\beta}\frac{t}{R})\partial_t\} + \frac{\alpha}{\beta}\frac{t}{R}(V\eta W + W\eta V) \\
&+ VW\eta\{t\partial_R + R\partial_t\} - R\frac{\alpha}{\beta}(V, W) + \nabla_V W
\end{aligned}$$

collecting up all the terms the result follows. □

In order to extract the extrinsic geometry of $\mathbb{S}^2_{\bar{r}}$ from Proposition A.1.1 we'll be needing an orthonormal basis $\{\nu, \nu^\star\}$ for the normal bundle. Recalling equations (A.4a) and (A.4b) we obtain a normal frame field,

$$\begin{aligned}
\{D(R - \bar{r}\cosh\eta), D(t - \bar{r}\sinh\eta)\} &= \{\frac{1}{\beta^4}\partial_R - \bar{r}\sinh\eta\frac{\nabla\eta}{R^2\beta^4}, -(\frac{\beta}{\alpha})^2\partial_t - \bar{r}\cosh\eta\frac{\nabla\eta}{R^2\beta^4}\} \\
&= \{\frac{1}{\beta^2}(\frac{\partial_R}{\beta^2}) - \frac{1}{\beta^2}\frac{t}{R}\frac{\nabla\eta}{R\beta^2}, -\frac{\beta}{\alpha}(\frac{\beta}{\alpha}\partial_t) - \frac{1}{\beta^2}\frac{\nabla\eta}{R\beta^2}\}
\end{aligned}$$

where $\nabla\eta := \mathrm{grad}_{\mathbb{S}^2}\eta$.

Thus, we may construct a normal vector field in $L(\mathbb{P})$ from the linear combination given by $D(R-\bar{r}\cosh\eta)-\frac{t}{R}D(t-\bar{r}\sinh\eta) = \frac{\beta}{\alpha}(\frac{\alpha}{\beta^3}(\frac{\partial_R}{\beta^2})+\frac{t}{R}(\frac{\beta}{\alpha}\partial_t))$. By denoting $\Gamma = (\frac{\alpha}{\beta^3})^2-(\frac{t}{R})^2$ and $\Lambda = 1-\Gamma|\nabla\eta|^2$ we find our orthonormal frame field from

Lemma A.1.1. *For sufficiently large $\bar{r}$, $\mathbb{S}^2_{\bar{r}}$ inherits an orthonormal frame field $\{\nu,\nu^\star\}$ given by:*

$$\Gamma^{\frac{1}{2}}\nu = \frac{\alpha}{\beta^3}(\frac{\partial_R}{\beta^2})+\frac{t}{R}(\frac{\beta}{\alpha}\partial_t)$$

$$\Gamma^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\nu^\star = \frac{t}{R}(\frac{\partial_R}{\beta^2})+\frac{\alpha}{\beta^3}(\frac{\beta}{\alpha}\partial_t)+\Gamma\frac{\nabla\eta}{R\beta^2}$$

Proof. It's clear that $\langle\nu^\star,\nu^\star\rangle = -1$ and $\langle\nu,\nu^\star\rangle = 0$ so it suffices to show that $\langle\nu^\star,\bar{V}\rangle = 0$ for any $V \in \eta(T\mathbb{S}^2)$. So we calculate,

$$\Gamma^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\langle\nu^\star,\bar{V}\rangle = V\eta\langle\frac{t}{R}(\frac{\partial_R}{\beta^2})+\frac{\alpha}{\beta^3}(\frac{\beta}{\alpha}\partial_t), t\partial_R+R\partial_t\rangle+\Gamma\langle\frac{\nabla\eta}{R\beta^2},V\rangle$$

$$= V\eta\{\frac{t^2}{R}\beta^2 - R\frac{\alpha}{\beta^3}\frac{\alpha}{\beta}\}+R\beta^2\Gamma(\nabla\eta,V)$$

$$= R\beta^2V\eta\{(\frac{t}{R})^2-(\frac{\alpha}{\beta^3})^2\}+R\beta^2\Gamma V\eta$$

$$= 0$$

□

Remark A.1.1. *Here 'sufficiently large $\bar{r}$' serves to ensure that $\Gamma > 0$ on $\mathbb{S}^2_{\bar{r}}$ and as a consequence $\Lambda > 0$.*

Denoting the second fundamental form of $\mathbb{S}^2_{\bar{r}}$ by $\mathrm{II}_{\bar{r}}$ we have,

Proposition A.1.2.

1. $\Gamma^{\frac{1}{2}}\langle\nu,\mathrm{II}_{\bar{r}}(V,W)\rangle = V\eta W\eta[R\frac{\alpha}{\beta}\{1-(\frac{t}{R})^2\}+\frac{M}{\beta^2}\{(\frac{\alpha}{\beta^3})^2-(\alpha+2)(\frac{t}{R})^2\}]$
$-R(\frac{\alpha}{\beta})^2(V,W)$

2. $\Gamma^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\langle\nu^\star, \mathrm{II}_{\bar r}(V,W)\rangle = V\eta W\eta[t\beta^2\{1-(\frac{\alpha}{\beta^3})^2\} + M\beta\frac{t}{R}\{(\frac{t}{R})^2 - (\frac{\alpha}{\beta^3})^2\frac{1+2\alpha}{\alpha}\}]$

$+ t\beta^2\{(\frac{R}{\bar r})^2\eta - \frac{\alpha}{\beta}\}(V,W)$

Proof.

1. Since $\langle D_{\bar V}\bar W, \nu\rangle = \langle \mathrm{II}_{\bar r}(V,W), \nu\rangle$ the result follows from Proposition A.1.1 and Lemma A.1.1 following a tedious yet straight forward calculation.

2. As in the proof of 1., after the use of Proposition A.1.1 and Lemma A.1.1 the expression eventually simplifies to,

$$\begin{aligned}\Gamma^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\langle\nu^\star, \mathrm{II}_{\bar r}(V,W)\rangle = V\eta W\eta[t\beta^2\{1-(\frac{\alpha}{\beta^3})^2\} - M\beta(\frac{t}{R})^3 - M\frac{\beta}{\alpha}(\frac{\alpha}{\beta^3})^2\frac{t}{R}]\\ - R\beta^2\Gamma V W\eta + R\beta^2\Gamma(\nabla_V W, \nabla\eta) - t\alpha\beta(V,W)\\ + 2t\alpha\beta\Gamma V\eta W\eta\end{aligned}$$

at which point the third term admits the substitution $(\nabla_V W, \nabla\eta) = V(W,\nabla\eta) - (W, \nabla_V\nabla\eta) = VW\eta - (W,\nabla_V\nabla\eta)$. This in turn removes $VW\eta$ from the expression and $(W,\nabla_V\nabla\eta)$ satisfies the following identity:

Lemma A.1.2. *As a function on round $\mathbb{S}^2$, the Hessian of η satisfies*

$$H^\eta_{\mathbb{S}^2}(V,W) = -\Big(\frac{tR}{\bar r^2}\circ f_{\bar r}\Big)(V,W) + 2\Big(\frac{t}{R}\circ f_{\bar r}\Big)V\eta W\eta.$$

Proof. From the identity $\nabla\eta = (\frac{R}{\bar r})^2\nabla(\frac{t}{R})$ we have $V\eta = (\frac{R}{\bar r})^2 V(\frac{t}{R})$ for any $V \in \Gamma(T\mathbb{S}^2)$ as well as

$$\begin{aligned}(W,\nabla_V\nabla\eta) &= W(\frac{t}{R})V((\frac{\bar r}{R})^{-2}) + (\frac{R}{\bar r})^2(W,\nabla_V\nabla(\frac{t}{R}))\\ &= W(\frac{t}{R})V((1-(\frac{t}{R})^2)^{-1}) + (\frac{R}{\bar r})^2(W,\nabla_V\nabla(\frac{t}{R}))\\ &= 2\frac{t}{R}(1-(\frac{t}{R})^2)^{-2}W(\frac{t}{R})V(\frac{t}{R}) + (\frac{R}{\bar r})^2(W,\nabla_V\nabla(\frac{t}{R}))\end{aligned}$$

$$= 2\frac{t}{R}(\frac{R}{\bar{r}})^4 W(\frac{t}{R})V(\frac{t}{R}) + (\frac{R}{\bar{r}})^2(W, \nabla_V\nabla(\frac{t}{R})).$$

So proving the Lemma is equivalent to showing that

$$(W, \nabla_V\nabla(\frac{t}{R})) = -\frac{t}{R}(V, W)$$

and since $\frac{t}{R} = \tanh\psi\cos\vartheta$ this is, in turn, equivalent to

$$H^{\cos\vartheta}_{\mathbb{S}^2} + \cos\vartheta\mathring{\gamma} \equiv 0$$

for $\mathring{\gamma}$ the standard round metric. This is easily verified for the basis $\{\partial_\vartheta, \partial_\varphi\}$. □

After substituting the identity of Lemma A.1.2 the expression simplifies to the result.

□

In order to find the mean curvature $\vec{H}$ of $\mathbb{S}^2_{\bar{r}}$ we will need the induced metric and its inverse. The metric we find from

$$\langle\bar{V}, \bar{W}\rangle = V\eta W\eta\langle X, X\rangle + \langle V, W\rangle = V\eta W\eta\{t^2\beta^4 - R^2(\frac{\alpha}{\beta})^2\} + \langle V, W\rangle$$

$$= R^2\beta^4\{(V, W) - \Gamma V\eta W\eta\}$$

So in a coordinate basis the induced metric and it's inverse take the form

$$g_{ij} = R^2\beta^4\{\mathring{\gamma}_{ij} - \Gamma\eta_i\eta_j\}$$

$$g^{ij} = \frac{1}{R^2\beta^4}\{\mathring{\gamma}^{ij} + \frac{\Gamma}{\Lambda}\eta^i\eta^j\}$$

for $\eta^i = \mathring{\gamma}^{ij}\eta_j$.

Corollary A.1.2.1.

1. $$R^2\beta^4\Gamma^{\frac{1}{2}}\Lambda\langle\nu,\vec{H}\rangle = -R(\frac{\alpha}{\beta})^2[2-\{\Gamma+\frac{\beta}{\alpha}(1-(\frac{t}{R})^2\}|\nabla\eta|^2]$$
$$+\frac{M}{\beta^2}\{(\frac{\alpha}{\beta^3})^2-(2+\alpha)(\frac{t}{R})^2\}|\nabla\eta|^2$$

2. $$R^2\beta^4\Gamma^{\frac{1}{2}}\Lambda^{\frac{3}{2}}\langle\nu^\star,\vec{H}\rangle = |\nabla\eta|^2[t\beta^2\{1-(\frac{\alpha}{\beta^3})^2\}+M\beta\frac{t}{R}\{(\frac{t}{R})^2-(\frac{\alpha}{\beta^3})^2\frac{1+2\alpha}{\alpha}\}]$$
$$+t\beta^2\{(\frac{R}{\bar{r}})^2\Gamma-\frac{\alpha}{\beta}\}(2-\Gamma|\nabla\eta|^2)$$

Proof. We make the following observations:

$$g^{ij}\eta_i\eta_j = \frac{1}{R^2\beta^4}(|\nabla\eta|^2+\frac{\Gamma}{\Lambda}|\nabla\eta|^4) = \frac{|\nabla\eta|^2}{\Lambda R^2\beta^4}$$
$$g^{ij}\mathring{\gamma}_{ij} = \frac{1}{R^2\beta^4}(2+\frac{\Gamma}{\Lambda}|\nabla\eta|^2) = \frac{2-\Gamma|\nabla\eta|^2}{\Lambda R^2\beta^4}$$

In either expression of Proposition A.1.1 taking a trace with g^{ij} yields the above factors. A straight forward simplification yields the result. □

Remark A.1.2. *We notice that* $\lim_{\bar{r}\to\infty}(R\beta^4\Gamma^{\frac{1}{2}}\langle\nu,\vec{H}\rangle) = -2$, $\lim_{\bar{r}\to\infty}\Gamma^{\frac{1}{2}} = \frac{\bar{r}}{R}$ *and* $\lim_{\bar{r}\to\infty}\beta = 1$. *As a result we deduce that* $\langle\nu,\vec{H}\rangle = -\frac{2}{\bar{r}}+O(\frac{1}{\bar{r}^2})$ *and similarly that* $\langle\nu^\star,\vec{H}\rangle = O(\frac{1}{\bar{r}^2})$. *This is to be expected since, for large* $\bar{r}$*, the spacelike normal* $R\nu$ *approaches the position vector field* $P = t\partial_t + R\partial_R$ *in Minkowski (or* $\mathbb{R}^4_1$*) and it's a well known fact that the coordinate sphere of radius* $\bar{r}$ *in any boosted* $\mathbb{R}^3\subset\mathbb{R}^4_1$ *has mean curvature* $\vec{H} = -\frac{2}{\bar{r}}\frac{P}{\bar{r}}$.

A.2 Curvature and Energy

Letting $\lambda = \lim_{\bar{r}\to\infty}\Lambda(\bar{r})$ we refine our estimate of $\langle\nu,\vec{H}\rangle$

Corollary A.2.0.1.

1. $\langle \nu, \vec{H} \rangle = -\frac{2}{\bar{r}} + \frac{f}{\bar{r}^2} + O(\frac{1}{\bar{r}^3})$, *where*

$$f = 2M\frac{R}{\bar{r}}\{(1+\lambda^{-1})(2(\frac{\bar{r}}{R})^2 + 1) - 4\}$$

2. $\langle \nu^\star, \vec{H} \rangle = \frac{g}{\bar{r}^2} + O(\frac{1}{\bar{r}^3})$, *where*

$$g = -2M\frac{t}{\bar{r}}\lambda^{-\frac{1}{2}}\{3 + \lambda^{-1} - (\frac{\bar{r}}{R})^2\}$$

Proof. 1. We start by approximating Γ,

$$\begin{aligned}\Gamma &= (\frac{\alpha}{\beta^3})^2 - (\frac{t}{R})^2 \\ &= (\frac{\alpha}{\beta^3})^2 - 1 + 1 - (\frac{t}{R})^2 \\ &= (\frac{\bar{r}}{R})^2\{1 - (\frac{R}{\bar{r}})^2(1 - (\frac{\alpha}{\beta^3})^2)\} \\ &= (\frac{\bar{r}}{R})^2\{1 - 4M\frac{R}{\bar{r}}\frac{1}{\bar{r}}\} + O(\frac{1}{\bar{r}^2})\end{aligned}$$

giving,

$$\Gamma^{\frac{1}{2}} = \frac{\bar{r}}{R}\{1 - 2M\frac{R}{\bar{r}}\frac{1}{\bar{r}}\} + O(\frac{1}{\bar{r}^2})$$

and,

$$R\Gamma^{\frac{1}{2}} = \bar{r}\{1 - 2M\frac{R}{\bar{r}}\frac{1}{\bar{r}}\} + O(\frac{1}{\bar{r}}).$$

From this we conclude that,

$$\begin{aligned}R\Gamma^{\frac{1}{2}}\langle \nu, \vec{H} \rangle &= (1 - 2M\frac{R}{\bar{r}}\frac{1}{\bar{r}})(-2 + \frac{f}{\bar{r}}) + O(\frac{1}{\bar{r}^2}) \\ &= -2 + (4M\frac{R}{\bar{r}} + f)\frac{1}{\bar{r}} + O(\frac{1}{\bar{r}^2}).\end{aligned}$$

Now,

$$\Lambda(2+R\Gamma^{\frac{1}{2}}\langle\nu,\vec{H}\rangle) = 2\Lambda - (\frac{\alpha}{\beta^3})^2[2-(\eta+\frac{\beta}{\alpha}(1-(\frac{t}{R})^2)|\nabla\eta|^2] + \frac{M}{R\beta^4}((\frac{\alpha}{\beta^3})^2$$

$$-(2+\alpha)(\frac{t}{R})^2)|\nabla\eta|^2$$

$$= 2\Big(1-(\frac{\alpha}{\beta^3})^2\Big) + (\frac{\alpha}{\beta^3})^2|\nabla\eta|^2\{\Big((\frac{\alpha}{\beta^3})^2-1\Big) + \Big(\frac{\beta}{\alpha}-1\Big)\}$$

$$+(\frac{t}{R})^2|\nabla\eta|^2\{\Big(1-(\frac{\alpha}{\beta^3})^2\Big) + \Big(1-\frac{\beta}{\alpha}(\frac{\alpha}{\beta^3})^2\Big)\}$$

$$+\frac{M}{R\beta^4}((\frac{\alpha}{\beta^3})^2-(2+\alpha)(\frac{t}{R})^2)|\nabla\eta|^2$$

where we've isolated with $\Big(\cdot\Big)$ all terms of magnitude $O(\frac{1}{\bar{r}})$. As $\bar{r}\to\infty$,

$$\lambda(4M\frac{R}{\bar{r}}+f) = \lim_{\bar{r}\to\infty}\Lambda\bar{r}(2+R\Gamma^{\frac{1}{2}}\langle\nu,\vec{H}\rangle)$$

$$= 8M\frac{\bar{r}}{R} + |\nabla\eta|^2\{-4M\frac{\bar{r}}{R}+M\frac{\bar{r}}{R}\} + |\nabla\eta|^2(\frac{t}{R})^2\{4M\frac{\bar{r}}{R}+3M\frac{\bar{r}}{R}\}$$

$$+M\frac{\bar{r}}{R}\{1-3(\frac{t}{R})^2\}|\nabla\eta|^2$$

$$= 8M\frac{\bar{r}}{R} - 2M\frac{\bar{r}}{R}(1-2(\frac{t}{R})^2)|\nabla\eta|^2$$

making the substitutions $|\nabla\eta|^2 = (\frac{R}{\bar{r}})^2(1-\lambda)$ and $(\frac{t}{R})^2 = 1-(\frac{\bar{r}}{R})^2$ the result follows.

2. From our approximation of Γ in 1. we see

$$(\frac{R}{\bar{r}})^2\Gamma - 1 = -4M\frac{R}{\bar{r}}\frac{1}{\bar{r}} + O(\frac{1}{\bar{r}^2})$$

so that

$$(\frac{R}{\bar{r}})^2\Gamma - \frac{\alpha}{\beta} = (\frac{R}{\bar{r}})^2\Gamma - 1 + \frac{\beta-\alpha}{\beta}$$

$$= M(\frac{\bar{r}}{R} - 4\frac{R}{\bar{r}})\frac{1}{\bar{r}} + O(\frac{1}{\bar{r}^2})$$

$$= M(1-4(\frac{R}{\bar{r}})^2)\frac{1}{R} + O(\frac{1}{\bar{r}^2}).$$

Using this we see directly from Corollary A.1.2.1

$$\frac{R}{\bar{r}}\lambda^{\frac{3}{2}} \lim_{\bar{r}\to\infty} \bar{r}^2\langle \nu^\star, \vec{H}\rangle = \lim_{\bar{r}\to\infty} R^2\beta^4\Gamma^{\frac{1}{2}}\Lambda^{\frac{3}{2}}\langle \nu^\star, \vec{H}\rangle$$

$$= |\nabla\eta|^2\{4M\frac{t}{R} + M\frac{t}{R}((\frac{t}{R})^2 - 3)\}$$

$$+ M\frac{t}{R}(1-4(\frac{R}{\bar{r}})^2)(2-(\frac{\bar{r}}{R})^2|\nabla\eta|^2)$$

$$= M\frac{t}{R}\{6|\nabla\eta|^2 + 2\lambda - 8(\frac{R}{\bar{r}})^2\}$$

$$= -2M\frac{t}{R}\{3(\frac{R}{\bar{r}})^2\lambda + (\frac{R}{\bar{r}})^2 - \lambda\}$$

$$= -2M\lambda\frac{tR}{\bar{r}^2}\{3 + \lambda^{-1} - (\frac{\bar{r}}{R})^2\}$$

□

Definition A.2.1. *The Hawking Energy of a closed surface* Σ *is given by;*

$$E_H = \sqrt{\frac{|\Sigma|}{16\pi}}\Big(1 - \frac{1}{16\pi}\int_\Sigma \langle \vec{H}, \vec{H}\rangle dA\Big)$$

It's a well known fact that the Hawking Energy for coordinate spheres in an asymptotically flat hypersurface approach the ADM energy as $\bar{r} \to \infty$. In Schwarzschild we see therefore that coordinate spheres in the cononical time slice $\{t = 0\}$ are round (of radius $\bar{r} = R\beta^2$) and satisfy $t = \eta = 0$ so that Corollary 2 gives us $\langle \vec{H}, \vec{H}\rangle = \langle \nu, \vec{H}\rangle^2 = \frac{4}{R^2}(\frac{\alpha}{\beta^3})^2$. Therefore

$$E_H = \sqrt{\frac{4\pi R^2\beta^4}{16\pi}}(1 - \frac{1}{16\pi}4\pi R^2\beta^4\frac{4}{R^2}(\frac{\alpha}{\beta^3})^2) = \frac{R\beta^2}{2}(1-(\frac{\alpha}{\beta})^2) = \frac{R}{2}(\alpha+\beta)(\alpha-\beta) = M.$$

As a result of boosting this slice to the rapidity ψ we expect the ADM energy to boost to $M\cosh\psi$. We are now in a position to verify this expectation.

Proposition A.2.1. *For the boosted time slice $\{\bar{t} = 0\}$ the Hawking Energy $E_H(\bar{r})$ of the coordinate sphere $\mathbb{S}^2_{\bar{r}}$ satisfies*

$$\lim_{\bar{r}\to\infty} E_H(\bar{r}) = M\cosh\psi$$

Proof. From the change $\vartheta \to \bar{\vartheta}$ given by,

$$\sin\bar{\vartheta} = \frac{\sin\vartheta}{\sqrt{1-\tanh^2\psi\cos^2\vartheta}}$$

$$\cos\bar{\vartheta} = \frac{\text{sech}\psi\cos\vartheta}{\sqrt{1-\tanh^2\psi\cos^2\vartheta}}$$

it's easily shown that

$$d\bar{\vartheta}^2 + (\sin\bar{\vartheta})^2 d\varphi^2 = (\frac{R}{\bar{r}})^2\{\lambda d\vartheta^2 + (\sin\vartheta)^2 d\varphi^2\} = \lim_{\bar{r}\to\infty}\frac{1}{\bar{r}^2} ds^2_{\mathbb{S}^2_{\bar{r}}}.$$

We conclude for large $\bar{r}$, $\mathbb{S}^2_{\bar{r}}$ approaches round $\mathbb{S}^2$ and therefore,

$$\frac{|\mathbb{S}^2_{\bar{r}}|}{\bar{r}^2} = \int\frac{\sqrt{\det g}}{\bar{r}^2}d\vartheta d\varphi = \int\Lambda^{\frac{1}{2}}(\frac{R}{\bar{r}})^2\beta^4\sin\vartheta d\vartheta d\varphi \xrightarrow{\bar{r}\to\infty}\int\lambda^{\frac{1}{2}}(\frac{R}{\bar{r}})^2\sin\vartheta d\vartheta d\varphi = 4\pi.$$

From our refined decomposition of $\vec{H}$

$$\langle\vec{H},\vec{H}\rangle = \Big(-\frac{2}{\bar{r}}+\frac{f}{\bar{r}^2}\Big)^2 + O(\frac{1}{\bar{r}^4}) = \frac{4}{\bar{r}^2} - 4\frac{f}{\bar{r}^3} + O(\frac{1}{\bar{r}^4})$$

we are able to approximate $E_H(\bar{r})$,

$$E_H(\bar{r}) = \frac{1}{4\pi}\sqrt{\frac{|\mathbb{S}^2_{\bar{r}}|}{16\pi\bar{r}^2}}\Big(\int(\frac{R}{\bar{r}})^2[\bar{r}(\lambda^{\frac{1}{2}} - \Lambda^{\frac{1}{2}}\beta^4) + f\Lambda^{\frac{1}{2}}\beta^4]\sin\vartheta d\vartheta d\varphi\Big) + O(\frac{1}{\bar{r}})$$

giving,

$$\lim_{\bar{r}\to\infty} E_H(\bar{r}) = \frac{1}{4\pi}\frac{1}{2}\int(\frac{R}{\bar{r}})^2[\lim_{\bar{r}\to\infty}(\bar{r}(\lambda^{\frac{1}{2}} - \Lambda^{\frac{1}{2}}\beta^4)) + f\lambda^{\frac{1}{2}}]\sin\vartheta\lambda\vartheta d\varphi.$$

So in order to calculate the limit, we will also need next to leading order information about $\sqrt{\det g}$. We find this next order term by noting that since

$$\Lambda = 1 - \Gamma|\nabla\eta|^2 = \lambda + (1 - (\frac{\alpha}{\beta^3})^2)|\nabla\eta|^2 = \lambda + \frac{4M}{R}|\nabla\eta|^2 + O(\frac{1}{\bar{r}^2})$$

we have that

$$\Lambda^{\frac{1}{2}}\beta^4 = \lambda^{\frac{1}{2}}(1 + \frac{2M}{\lambda R}|\nabla\eta|^2)(1 + \frac{M}{2R})^4 + O(\frac{1}{\bar{r}^2}) = \lambda^{\frac{1}{2}} + \frac{2M}{R}(\lambda^{\frac{1}{2}} + \frac{|\nabla\eta|^2}{\lambda^{\frac{1}{2}}}) + O(\frac{1}{\bar{r}^2})$$

giving

$$\lim_{\bar{r}\to\infty}(\bar{r}(\lambda^{\frac{1}{2}} - \Lambda^{\frac{1}{2}}\beta^4)) = -2M\frac{\bar{r}}{R}\lambda^{\frac{1}{2}}(1 + \frac{|\nabla\eta|^2}{\lambda}).$$

so that

$$\lim_{\bar{r}\to\infty} E_H(\bar{r}) = \frac{1}{8\pi}\int_{\mathbb{S}^2} f - 2M\frac{\bar{r}}{R}(1 + \frac{|\nabla\eta|^2}{\lambda})dA.$$

Using Corollary A.2.0.1 it's an easy calculation to show that

$$\begin{aligned}\{f - 2M\frac{\bar{r}}{R}(1 + \frac{|\nabla\eta|^2}{\lambda})\}\lambda^{\frac{1}{2}}(\frac{R}{\bar{r}})^2 &= 2M\{\frac{2}{\lambda^{\frac{1}{2}}}\frac{R}{\bar{r}} + \lambda^{\frac{1}{2}}\frac{\bar{r}}{R}((\frac{R}{\bar{r}})^2 - 2(\frac{R}{\bar{r}})^4)\}\\ &= 2M\{2\cosh\psi + \operatorname{sech}\psi((\frac{R}{\bar{r}})^2 - 2(\frac{R}{\bar{r}})^4)\}\end{aligned}$$

so that finally we calculate

$$\begin{aligned}\lim_{\bar{r}\to\infty} E_H(\bar{r}) &= \frac{M}{4\pi}\int_0^{\pi}\int_0^{2\pi}\{2\cosh\psi + \operatorname{sech}\psi((\frac{R}{\bar{r}})^2 - 2(\frac{R}{\bar{r}})^4)\}\sin\vartheta\, d\vartheta d\varphi\\ &= 2M\cosh\psi + \frac{M}{2}\operatorname{sech}\psi\int_{-1}^{1}\frac{1}{1-\tanh^2\psi x^2} - 2\frac{1}{(1-\tanh^2\psi x^2)^2}dx\\ &= 2M\cosh\psi + \frac{M}{2}\operatorname{sech}\psi\Big[-(1 + c\frac{\partial}{\partial c})\int_{-1}^{1}\frac{1}{1-c^2x^2}dx\Big]\Big|_{c=\tanh\psi}\\ &= 2M\cosh\psi + \frac{M}{2}\operatorname{sech}\psi\Big[-\frac{2}{c}\tanh^{-1}c - c(-\frac{2}{c^2}\tanh^{-1}c + \frac{2}{c}\frac{1}{1-c^2})\Big]\Big|_{c=\tanh\psi}\\ &= M\cosh\psi.\end{aligned}$$

□

We will now work towards a stronger result from which we can deduce Proposition A.2.1. Specifically, we calculate $K_{\mathbb{S}^2_{\bar{r}}} - \frac{1}{4}\langle \vec{H}, \vec{H}\rangle$ from the Gauss equation:

Lemma A.2.1. *In the case of Schwarzschild, $\mathbb{P} \times_{R\beta^2} \mathbb{S}^2$, we have,*

1. $\langle R_{\partial_R \partial_t}\partial_R, \partial_t\rangle = -\frac{2M}{R^2}\frac{1}{R\beta^2}(\frac{\alpha}{\beta})^2$

2. (a) $H^{R\beta^2}(\partial_R, \partial_R) = \frac{M}{R^2}$

   (b) $H^{R\beta^2}(\partial_R, \partial_t) = 0$

   (c) $H^{R\beta^2}(\partial_t, \partial_t) = -\frac{M}{R^2}(\frac{\alpha}{\beta^3})^2$

Proof. Using Corollary A.0.1.1 thoughout:

1. $$\begin{aligned}\langle R_{\partial_R \partial_t}\partial_R, \partial_t\rangle &= \langle D_{\partial_t} D_{\partial_R}\partial_R, \partial_t\rangle - \langle D_{\partial_R} D_{\partial_t}\partial_R, \partial_t\rangle \\ &= \partial_t\langle D_{\partial_R}\partial_R, \partial_t\rangle - \langle D_{\partial_R}\partial_R, D_{\partial_t}\partial_t\rangle - \partial_R\langle D_{\partial_t}\partial_R, \partial_t\rangle + \langle D_{\partial_t}\partial_R, D_{\partial_R}\partial_t\rangle \\ &= 0 + \frac{M}{R^2\beta}\frac{\alpha}{\beta^3}\frac{1}{\beta^4}\frac{M}{R^2}\langle\partial_R, \partial_R\rangle - \partial_R(\frac{M}{\alpha\beta R^2}\langle\partial_t, \partial_t\rangle) + (\frac{M}{\alpha\beta R^2})^2\langle\partial_t, \partial_t\rangle \\ &= \partial_R(\frac{M\alpha}{R^2\beta^3}) + \frac{M^2\alpha}{R^4\beta^4} - \frac{M^2}{R^4\beta^4} \\ &= -\frac{2M\alpha}{R^3\beta^3} + \frac{M}{R^2}(\partial_R(\frac{\alpha}{\beta^3}) - M\frac{1+\alpha}{R^2\beta^4}) + \frac{2M^2\alpha}{R^4\beta^4} \\ &= -\frac{2M\alpha}{R^3\beta^3}(1 - \frac{M}{R\beta}) \\ &= -\frac{2M\alpha^2}{R^3\beta^4}\end{aligned}$$

   where we used the fact that $\partial_R(\frac{\alpha}{\beta^3}) = \frac{M}{R^2}\frac{1+\alpha}{\beta^4}$ in the sixth line.

2. (a) $\langle \partial_R, D_{\partial_R} D(R\beta^2)\rangle = \partial_R^2(R\beta^2) - \langle D_{\partial_R}\partial_R, D(R\beta^2)\rangle$

$$= \partial_R^2(R\beta^2) + \frac{M}{R^2\beta}\partial_R(R\beta^2)$$

$$= \partial_R(\beta^2 - \beta\frac{M}{R}) + \frac{M}{R^2\beta}(\beta^2 - \beta\frac{M}{R}) = \partial_R(\alpha\beta) + \frac{M\alpha}{R^2}$$

$$= \alpha\partial_R\beta + \beta\partial_R\alpha + \frac{M\alpha}{R^2} = \frac{M}{R^2}(\frac{\beta-\alpha}{2} + \alpha)$$

$$= \frac{M}{R^2}$$

(b) $\langle \partial_R, D_{\partial_t} D(R\beta^2)\rangle = \frac{\partial_R(R\beta^2)}{\beta^4}\frac{1}{2}\partial_t\langle\partial_R, \partial_R\rangle = 0$

(c) $\langle \partial_t, D_{\partial_t} D(R\beta^2)\rangle = \dfrac{\partial_R(R\beta^2)}{\beta^4}\langle\partial_t, D_{\partial_t}\partial_R\rangle = \dfrac{\alpha\beta}{\beta^4}\dfrac{1}{2}\partial_R\langle\partial_t, \partial_t\rangle$

$$= \frac{\alpha}{\beta^3}(-\frac{\alpha}{\beta^3}\frac{m}{R^2}).$$

□

Proposition A.2.2. *Let $M = B \times_f F$ be a warped product with Riemannian curvature tensor R. If $X, Y, Z \in L(B)$ and $U, V, W \in L(F)$, then*

1. *$R_{XY}Z \in L(B)$ is the lift of ${}^B R_{XY}Z$ on B.*

2. *$R_{VX}Y = \frac{H^f(X,Y)}{f}V$, where H^f is the Hessian of f.*

3. *$R_{XY}V = R_{VW}X = 0$*

4. *$R_{XV}W = \frac{\langle V,W\rangle}{f}D_X Df$*

5. *$R_{VW}U = {}^F R_{VW}U - \frac{\langle Df,Df\rangle}{f^2}\{\langle V,U\rangle W - \langle W,U\rangle V\}$*

Proof. see [21] pg.210. □

Proposition A.2.3. *For $\Sigma = \mathbb{S}^2_{\bar{r}}$*

$$K_\Sigma - \frac{1}{4}\langle\vec{H},\vec{H}\rangle + \frac{1}{2}\langle\hat{\text{I}}\text{I}, \hat{\text{I}}\text{I}\rangle = \frac{M}{\Lambda(R\beta^2)^3}(2 + \Gamma|\nabla\eta|^2)$$

Proof. Knowing that $G(\cdot,\cdot)$ vanishes identically in Schwarzschild our task is to calculate the quantity $\langle R_{\nu\nu^\star}\nu,\nu^\star\rangle = \frac{1}{4}\langle R_{\underline{L}L}\underline{L},L\rangle$ (for $\underline{L}=\nu-\nu^\star$ and $L=\nu+\nu^\star$) so that the result follows from the Gauss equation (Proposition 3.0.1). We use our standard choice for $\{\nu,\nu^\star\}$ and break $\nu^\star$ into the components $\Lambda^{\frac{1}{2}}\nu^\star = \nu_1^\star + \nu_2$ where:

$$\nu_1^\star = \frac{\frac{t}{R}[\frac{\partial_R}{\beta^2}] + \frac{\alpha}{\beta^3}[\frac{\beta}{\alpha}\partial_t]}{\Gamma^{\frac{1}{2}}}$$

and

$$\nu_2 = \Gamma^{\frac{1}{2}}\frac{\nabla\eta}{R\beta^2}$$

(we temporarily denote also $\nu_1=\nu$).

We will need to find the determinant of the linear map

$$\begin{pmatrix} \frac{t}{R}\frac{1}{\beta^2} & \frac{\alpha}{\beta^3}\frac{\beta}{\alpha} \\ \frac{\alpha}{\beta^3}\frac{1}{\beta^2} & \frac{t}{R}\frac{\beta}{\alpha}\end{pmatrix}\begin{pmatrix}\partial_t\\ \partial_R\end{pmatrix} = \Gamma^{\frac{1}{2}}\begin{pmatrix}\nu_1^\star\\ \nu_1\end{pmatrix}$$

which is easily seen to give

$$\frac{(\frac{t}{R})^2\frac{1}{\alpha\beta} - (\frac{\alpha}{\beta^3})^2\frac{1}{\alpha\beta}}{\Gamma} = -\frac{1}{\alpha\beta}.$$

Thus Proposition A.2.2 gives

$$\begin{aligned}
\Lambda\langle R_{\nu\nu^\star}\nu,\nu^\star\rangle &= \langle R_{\nu_1\,\nu_1^\star+\nu_2}\nu_1,\nu_1^\star+\nu_2\rangle = \langle R_{\nu_1\,\nu_1^\star}\nu_1,\nu_1^\star\rangle - \langle R_{\nu_2\,\nu_1}\nu_1,\nu_2\rangle \\
&= \frac{1}{\alpha^2\beta^2}\langle R_{\partial_R\partial_t}\partial_R,\partial_t\rangle - \frac{\langle\nu_2,\nu_2\rangle}{R\beta^2}H^{R\beta^2}(\nu_1,\nu_1) \\
&= -\frac{2M}{(R\beta^2)^3} - \frac{\Gamma|\nabla\eta|^2}{R\beta^2}[\frac{1}{\Gamma}\{(\frac{\alpha}{\beta^3})^2\frac{1}{\beta^4}H^{R\beta^2}(\partial_R,\partial_R) + (\frac{t}{R})^2(\frac{\beta}{\alpha})^2H^{R\beta^2}(\partial_t,\partial_t)\}] \\
&= -\frac{2M}{(R\beta^2)^3} - \frac{|\nabla\eta|^2}{R\beta^2}[(\frac{\alpha}{\beta^3})^2\frac{1}{\beta^4}\frac{M}{R^2} - (\frac{t}{R})^2(\frac{\beta}{\alpha})^2\frac{M}{R^2}(\frac{\alpha}{\beta^3})^2] \\
&= -\frac{2M}{(R\beta^2)^3} - \frac{|\nabla\eta|^2}{R\beta^2}(\frac{\alpha}{\beta^3})^2\frac{M}{R^2}(\frac{\beta}{\alpha})^2\Gamma
\end{aligned}$$

$$= -\frac{M}{(R\beta^2)^3}(2 + \Gamma|\nabla\eta|^2)$$

Having used Lemma A.2.1 in the forth. □

Lemma A.2.2. $\mathbb{S}^2_{\bar{r}}$ *satisfies*

$$\langle \hat{\mathrm{II}}, \hat{\mathrm{II}} \rangle = O(\frac{1}{\bar{r}^4}).$$

Proof. From Proposition A.1.2 and Corollary A.1.2.1 we conclude that the quantities

$$\langle \hat{\mathrm{II}}(V, W), \nu \rangle = \langle \mathrm{II}(V, W), \nu \rangle - \frac{1}{2}\langle V, W \rangle \langle \vec{H}, \nu \rangle$$

$$\langle \hat{\mathrm{II}}(V, W), \nu^\star \rangle = \langle \mathrm{II}(V, W), \nu^\star \rangle - \frac{1}{2}\langle V, W \rangle \langle \vec{H}, \nu^\star \rangle$$

converge as $\bar{r} \to \infty$ for any $V, W \in \Gamma(T\mathbb{S}^2)$. Therefore, from the choice of orthonormal frame

$$\{e_1 = \frac{\partial_\vartheta}{\Lambda^{\frac{1}{2}} R\beta^2},\ e_2 = \frac{\partial_\varphi}{\sin\vartheta R\beta^2}\}$$

we conclude that

$$\langle \hat{\mathrm{II}}, \hat{\mathrm{II}} \rangle = \sum_{i,j} \Big(\langle \hat{\mathrm{II}}(e_i, e_j), \nu \rangle^2 - \langle \hat{\mathrm{II}}(e_i, e_j), \nu^\star \rangle^2 \Big) = O(\frac{1}{\bar{r}^4})$$

□

Proposition A.2.2 therefore allows us to directly conclude that

$$\lim_{\bar{r}\to\infty} \bar{r}^3 \{K_\Sigma - \frac{1}{4}\langle \vec{H}, \vec{H} \rangle + \frac{1}{2}\langle \hat{\mathrm{II}}, \hat{\mathrm{II}} \rangle\} = \frac{M}{\lambda}(\frac{\bar{r}}{R})^3(2 + (\frac{\bar{r}}{R})^2|\nabla\eta|^2).$$

Moreover, from Lemma A.2.2

$$\lim_{\bar{r}\to\infty} E_H(\bar{r}) = \lim_{\bar{r}\to\infty} \frac{1}{4\pi}\sqrt{\frac{|\mathbb{S}^2_{\bar{r}}|}{16\pi}} \int_{\mathbb{S}^2_{\bar{r}}} K_{\mathbb{S}^2_{\bar{r}}} - \frac{1}{4}\langle \vec{H}, \vec{H} \rangle + \frac{1}{2}\langle \hat{\mathrm{II}}, \hat{\mathrm{II}} \rangle dA$$

$$= \frac{1}{4}\int_0^\pi \frac{M}{\lambda^{\frac{1}{2}}} \frac{\bar{r}}{R}(2 + (\frac{\bar{r}}{R})^2|\nabla\eta|^2)\sin\vartheta d\vartheta$$

$$= \frac{1}{4} M \cosh\psi \int_0^\pi (1 - \tanh^2\psi \cos^2\vartheta)(2 + \frac{\tanh^2\psi \sin^2\vartheta}{1 - \tanh^2\psi\cos^2\vartheta}) \sin\vartheta d\vartheta$$

$$= \frac{1}{4} M \cosh\psi \int_{-1}^1 2 - 2\tanh^2\psi x^2 + \tanh^2\psi(1 - x^2)dx$$

$$= \frac{1}{4} M \cosh\psi (4 - \frac{4}{3}\tanh^2\psi + \tanh^2\psi(2 - \frac{2}{3}))$$

$$= M\cosh\psi$$

as expected.

A.3 Normal Connection and Momentum

Finally, we finish our study of $\mathbb{S}^2_{\bar{r}}$ with its connection 1-form. From this we are able to deduce the ADM-momentum of our boosted slice.

Definition A.3.1. *Given a normal frame field $\{\nu, \nu^\star\}$ for $\mathbb{S}^2_{\bar{r}}$ and $V \in \Gamma(T\mathbb{S}^2)$, the connection 1-form associated to this frame is given by:*

$$\alpha_\nu(V) = \langle D_{\bar{V}}\nu, \nu^\star \rangle.$$

Proposition A.3.1. *In our standard normal frame $\{\nu, \nu^\star\}$ we have,*

$$\Gamma\Lambda^{\frac{1}{2}}\alpha_\nu(V) = V\eta[\frac{\alpha}{\beta^3}\{\frac{\alpha}{\beta}(\frac{\alpha}{\beta^3})^2 - 1\} + \frac{M}{R}\frac{1}{\beta^4}\{2(1+\alpha)(\frac{t}{R})^2 - (\frac{\alpha}{\beta^3})^2\}]$$

Proof.

$$D_{\bar{V}}(\Gamma^{\frac{1}{2}}\nu) = V\eta D_X(\Gamma^{\frac{1}{2}}\nu) + D_V(\Gamma^{\frac{1}{2}}\nu)$$

$$= V\eta\{tD_{\partial_R}(\Gamma^{\frac{1}{2}}\nu) + RD_{\partial_t}(\Gamma^{\frac{1}{2}}\nu)\} + \frac{\Gamma^{\frac{1}{2}}\nu(R\beta^2)}{R\beta^2}V$$

Now

1. $tD_{\partial_R}(\Gamma^{\frac{1}{2}}\nu) = \frac{t}{R}\{\frac{M}{R}\frac{1+\alpha}{\beta^6}\partial_R - \frac{t}{R}\frac{\beta}{\alpha}\partial_t\}$:

$$D_{\partial_R}(\Gamma^{\frac{1}{2}}\nu) = \partial_R(\frac{\alpha}{\beta^3}\frac{1}{\beta^2})\partial_R + \frac{\alpha}{\beta^3}\frac{1}{\beta^2}D_{\partial_R}\partial_R + \partial_R(\frac{t}{R}\frac{\beta}{\alpha})\partial_t + \frac{t}{R}\frac{\beta}{\alpha}D_{\partial_R}\partial_t$$

$$= \{\frac{M}{R^2}\frac{1+\alpha}{\beta^4}\frac{1}{\beta^2} + \frac{\alpha}{\beta^3}\frac{M}{R^2}\frac{1}{\beta^3}\}\partial_R - \frac{\alpha}{\beta^3}\frac{1}{\beta^2}\frac{M}{R^2}\frac{1}{\beta}\partial_R$$

$$+ \{-\frac{t}{R^2}\frac{\beta}{\alpha} - \frac{t}{R}\frac{M}{R^2}\frac{1}{\alpha^2}\}\partial_t + \frac{t}{R}\frac{\beta}{\alpha}\frac{M}{R^2}\frac{1}{\alpha\beta}\partial_t$$

$$= \frac{M}{R^2}\frac{1+\alpha}{\beta^6}\partial_R - \frac{t}{R^2}\frac{\beta}{\alpha}\partial_t$$

2. $RD_{\partial_t}(\Gamma^{\frac{1}{2}}\nu) = (\frac{\beta}{\alpha} + \frac{M}{R}\frac{1}{\beta^6})\partial_t + \frac{M}{R}\frac{t}{R}\frac{1}{\beta^6}\partial_R$:

$$D_{\partial_t}(\Gamma^{\frac{1}{2}}\nu) = \frac{\alpha}{\beta^3}\frac{1}{\beta^2}D_{\partial_t}\partial_R + \frac{1}{R}\frac{\beta}{\alpha}\partial_t + \frac{t}{R}\frac{\beta}{\alpha}D_{\partial_t}\partial_t$$

$$= \{\frac{\alpha}{\beta^3}\frac{1}{\beta^2}\frac{M}{R^2}\frac{1}{\alpha\beta} + \frac{1}{R}\frac{\beta}{\alpha}\}\partial_t + \frac{t}{R}\frac{\beta}{\alpha}\frac{\alpha}{\beta^3}\frac{1}{\beta^4}\frac{M}{R^2}\partial_R$$

3. $\dfrac{\Gamma^{\frac{1}{2}}\nu(R\beta^2)}{R\beta^2} = \dfrac{1}{R}(\dfrac{\alpha}{\beta^3})^2$:

$$\frac{\Gamma^{\frac{1}{2}}\nu(R\beta^2)}{R\beta^2} = \frac{\alpha}{\beta^3}\frac{1}{\beta^2}\frac{\partial_R(R\beta^2)}{R\beta^2} = \frac{\alpha}{\beta^3}\frac{1}{\beta^2}(\frac{1}{R} - \frac{M}{R^2}\frac{1}{\beta})$$

Giving

$$D_{\bar{V}}(\Gamma^{\frac{1}{2}}\nu) = V\eta[\{\frac{\beta}{\alpha}(1-(\frac{t}{R})^2) + \frac{M}{R}\frac{1}{\beta^6}\}\partial_t + \frac{t}{R}\frac{M}{R}\frac{2+\alpha}{\beta^6}\partial_R] + \frac{1}{R}(\frac{\alpha}{\beta^3})^2 V.$$

As a result,

$$\langle D_{\bar{V}}(\Gamma^{\frac{1}{2}}\nu), \Gamma^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\nu^\star\rangle = \Gamma\Lambda^{\frac{1}{2}}\langle D_{\bar{V}}\nu, \nu^\star\rangle$$

$$= V\eta\langle\{\frac{\beta}{\alpha}(1-(\frac{t}{R})^2) + \frac{M}{R}\frac{1}{\beta^6}\}\partial_t + \frac{t}{R}\frac{M}{R}\frac{2+\alpha}{\beta^6}\partial_R, \frac{t}{R}(\frac{\partial_R}{\beta^2}) + \frac{\alpha}{\beta^3}(\frac{\beta}{\alpha}\partial_t)\rangle$$

$$+ V\eta\frac{\alpha^2}{\beta^4}\Gamma$$

$$= V\eta[\frac{\alpha}{\beta^3}\{\frac{\alpha}{\beta}\Gamma - (1-(\frac{t}{R})^2)\} + \frac{M}{R}\{(\frac{t}{R})^2\frac{2+\alpha}{\beta^4} - (\frac{\alpha}{\beta^3})^2\frac{1}{\beta^4}\}]$$

$$= V\eta[\frac{\alpha}{\beta^3}\{\frac{\alpha}{\beta}(\frac{\alpha}{\beta^3})^2 - 1\} + \frac{M}{R}\frac{1}{\beta^4}\{2(1+\alpha)(\frac{t}{R})^2 - (\frac{\alpha}{\beta^3})^2\}]$$

□

It now easily follows that

Corollary A.3.1.1. *$\alpha_\nu(V) = V\eta\{\frac{h}{\bar{r}} + O(\frac{1}{\bar{r}^2})\}$ for,*

$$h = -2M\frac{R}{\bar{r}}\lambda^{-\frac{1}{2}}\{1 + 2(\frac{\bar{r}}{R})^2\}$$

Proof. □

In order to get hold of the second fundamental from K of our slice $\{\bar{t} = 0\}$ we will need to find the unit timelike normal $\vec{N}\propto D\bar{t}$ which we now work towards.

Lemma A.3.1.

$$D\bar{t} = -\cosh\psi\{\frac{t}{R}D(R - \bar{r}\cosh\eta) - D(t - \bar{r}\sinh\eta)\}$$
$$= -\cosh\psi\{\frac{t}{R}\frac{1}{\beta^2}(\frac{\partial_R}{\beta^2}) + \frac{\beta}{\alpha}(\frac{\beta}{\alpha}\partial_t) + \frac{1}{\beta^2}(1 - (\frac{t}{R})^2)\frac{\nabla\eta}{R\beta^2}\}$$

Proof. We recall that $\bar{t} = \cosh\psi t - \sinh\psi z = \cosh\psi t - \sinh\psi R\cos\vartheta$ so that,

$$D\bar{t} = \cosh\psi\{-(\frac{\beta}{\alpha})^2\partial_t - \tanh\psi(\frac{1}{\beta^4}\cos\vartheta\partial_R - \frac{R}{R^2\beta^4}\sin\vartheta\partial_\vartheta)\}.$$

We also recall that $\nabla\eta = \dfrac{-\tanh\psi\sin\vartheta}{1 - \tanh^2\psi\cos^2\vartheta}\partial_\vartheta$ allowing for the above expression to be written as

$$D\bar{t} = -\cosh\psi\{(\frac{\beta}{\alpha})^2\partial_t + \frac{1}{\beta^4}\frac{t}{R}\partial_R + \frac{R}{R^2\beta^4}(1 - (\frac{t}{R})^2)\nabla\eta\}$$

and the result follows. □

Remark A.3.1. *We notice that* $\langle D\bar{t}, D\bar{t}\rangle = \cosh^2\psi\{(\frac{t}{R})^2\frac{1}{\beta^4} - (\frac{\beta}{\alpha})^2 + \frac{1}{\beta^4}(1 - (\frac{t}{R})^2)|\nabla\eta|^2\}$ *has the same limit for large $\bar{r}$ as* $-\cosh^2\psi\Gamma\Lambda$.

Since $D\bar{t}$ is the timelike normal to our boosted slice $\{\bar{t} = 0\}$ and therefore normal to $\mathbb{S}^2_{\bar{r}}$ we must have some function $\varphi_n \in \mathcal{F}(\mathbb{S}^2)$ such that,

$$\vec{N} := -\frac{D\bar{t}}{|D\bar{t}|} = \cosh\phi_n \nu^\star + \sinh\phi_n \nu.$$

It's an easy calculation to show that

$$\sinh\phi_n = \langle\vec{N}, \nu\rangle = -\frac{\langle D\bar{t}, \Gamma^{\frac{1}{2}}\nu\rangle}{|D\bar{t}|\Gamma^{\frac{1}{2}}} = \frac{t}{R}\frac{\beta}{\alpha}\frac{(\frac{\alpha}{\beta^3})^2 - 1}{|D\bar{t}|\Gamma^{\frac{1}{2}}}$$

which has magnitude $O(\frac{1}{\bar{r}})$ indicating that $\vec{N}$ approaches $\nu^\star$ for large $\bar{r}$. Since $\lim_{\bar{r}\to\infty}\frac{\sinh\phi_n}{\phi_n} = 1$:

$$\lim_{\bar{r}\to\infty}\bar{r}\varphi_n = -\frac{t}{R}\lim_{\bar{r}\to\infty}\frac{\bar{r}(1 - (\frac{\alpha}{\beta^3})^2)}{|D\bar{t}|\eta^{\frac{1}{2}}} = -\frac{t}{R}\frac{4M\frac{\bar{r}}{R}}{(\frac{\bar{r}}{R})^2\lambda^{\frac{1}{2}}}$$

or equivalently,

Lemma A.3.2. *Given $\phi_n \in \mathcal{F}(\mathbb{S}^2)$ such that $\vec{N} = \cosh\phi_n\nu^\star + \sinh\phi_n\nu$ we have, $\varphi_n = \frac{k}{\bar{r}} + O(\frac{1}{\bar{r}^2})$ for,*

$$k = -4m\frac{t}{\bar{r}}d^{-\frac{1}{2}}.$$

Proof. □

Corollary A.3.1.2. *For the orthonormal frame field $\{\vec{N}, \vec{n}\}$ in the normal bundle of $\mathbb{S}^2_{\bar{r}}$ such that $\langle\partial_{\bar{r}}, \vec{n}\rangle > 0$:*

1. *$\langle\vec{N}, \vec{H}\rangle = \frac{l}{\bar{r}^2} + O(\frac{1}{\bar{r}^3})$ for,*

$$l = 2M\frac{t}{\bar{r}}\lambda^{-\frac{1}{2}}\{1 - \lambda^{-1} + (\frac{\bar{r}}{R})^2\}$$

2. $\alpha_{\vec{n}}(V) = V\eta\{\frac{s}{\bar{r}} + O(\frac{1}{\bar{r}^2})\}$ *for,*

$$s = -2m\frac{R}{\bar{r}}\lambda^{-\frac{1}{2}}$$

Proof. 1. Since

$$\langle\vec{N},\vec{H}\rangle = \cosh\phi_n\langle\nu^\star,\vec{H}\rangle + \sinh\phi_n\langle\nu,\vec{H}\rangle$$

according to Corollary A.2.0.1 and Lemma A.3.2,

$$\begin{aligned}\lim_{\bar{r}\to\infty}\bar{r}^2\langle\vec{N},\vec{H}\rangle &= -2M\frac{t}{\bar{r}}\lambda^{-\frac{1}{2}}\{3+\lambda^{-1}-(\frac{\bar{r}}{R})^2\} - 4M\frac{t}{\bar{r}}\lambda^{-\frac{1}{2}}(-2)\\ &= 2M\frac{t}{\bar{r}}\lambda^{-\frac{1}{2}}\{1-\lambda^{-1}+(\frac{\bar{r}}{R})^2\}.\end{aligned}$$

2. From

$$\vec{n} = \cosh\phi_n\nu + \sinh\phi_n\nu^\star$$

it's an easy exercise to show that $\alpha_{\vec{n}} = \alpha_\nu - d\phi_n$ giving

$$\begin{aligned}\lim_{\bar{r}\to\infty}\bar{r}\alpha_{\vec{n}}(V) &= V\eta[-2M\frac{R}{\bar{r}}\lambda^{-\frac{1}{2}}\{1+2(\frac{\bar{r}}{R})^2\}] - V(-4M\frac{t}{\bar{r}}\lambda^{-\frac{1}{2}})\\ &= (\frac{R}{\bar{r}})^2V(\frac{t}{R})[-2M\frac{R}{\bar{r}}\lambda^{-\frac{1}{2}}\{1+2(\frac{\bar{r}}{R})^2\}] + 4MV(\frac{t}{R}\frac{R}{\bar{r}}\lambda^{-\frac{1}{2}})\\ &= V(\frac{t}{R})[-2M(\frac{R}{\bar{r}})^3\lambda^{-\frac{1}{2}}\{1+2(\frac{R}{\bar{r}})^2\} + 4M\frac{R}{\bar{r}}\lambda^{-\frac{1}{2}}]\\ &= V(\frac{t}{R})(2M\frac{R}{\bar{r}}\lambda^{-\frac{1}{2}})[2-\{(\frac{R}{\bar{r}})^2+2\}]\\ &= -2M\frac{R}{\bar{r}}\lambda^{-\frac{1}{2}}V\eta\end{aligned}$$

having used the fact that $\cosh\psi = \frac{R}{\bar{r}}\lambda^{-\frac{1}{2}}$ in the second line.

□

We are now in a position to verify that the ADM-momentum of our boosted slice is given by $\vec{P} = (0,0,M\sinh\psi)$,

Proposition A.3.2. *In the boosted isotropic coordinates* $(x^1, x^2, x^3) = (\bar{x}, \bar{y}, \bar{z})$,

$$P_i := \lim_{\bar{r}\to\infty} \frac{1}{8\pi} \int_{\mathbb{S}^2_{\bar{r}}} \{K_{ij} - K\bar{g}_{ij}\}\vec{n}^j dA = M \sinh\psi \delta_{i3}$$

where $K(\cdot,\cdot) = -\langle \mathrm{II}_{\bar{t}}(\cdot,\cdot), \vec{N}\rangle$ *is the scalar second fundamental form of our boosted slice* ($K = \bar{g}^{ij}K_{ij}$) *and* $\vec{n}$ *is the outward unit normal to* $\mathbb{S}^2_{\bar{r}}$.

Proof. From the induced metric on our boosted slice we read off,

$$\partial_{\bar{x}} = \beta^4 D\bar{x}$$

$$\partial_{\bar{y}} = \beta^4 D\bar{y}$$

$$\partial_{\bar{z}} = \beta^4\{\cosh^2\psi - (\frac{\alpha}{\beta^3})^2 \sinh^2\psi\}D\bar{z}$$

giving

$$\begin{aligned} |D(\bar{r}^2)|\vec{n} &= D(\bar{r}^2) \\ &= 2\bar{x}D\bar{x} + 2\bar{y}D\bar{y} + 2\bar{z}D\bar{z} \\ &= \frac{2\bar{x}}{\beta^4}\partial_{\bar{x}} + \frac{2\bar{y}}{\beta^4}\partial_{\bar{y}} + \frac{2\bar{z}}{\cosh^2\psi - (\frac{\alpha}{\beta^3})^2\sinh^2\psi}\partial_{\bar{z}}. \end{aligned}$$

Thus, under restriction to $\mathbb{S}^2_{\bar{r}}$ we see $|D(\bar{r}^2)| = 2\bar{r} + O(1)$ so that,

$$\langle \vec{n}, \partial_{\bar{x}}\rangle = \frac{\bar{x}}{\bar{r}} + O(\frac{1}{\bar{r}}) = \frac{R}{\bar{r}}\sin\vartheta\cos\varphi + O(\frac{1}{\bar{r}}) = \sin\bar{\vartheta}\cos\varphi + O(\frac{1}{\bar{r}})$$

$$\langle \vec{n}, \partial_{\bar{y}}\rangle = \frac{\bar{y}}{\bar{r}} + O(\frac{1}{\bar{r}}) = \frac{R}{\bar{r}}\sin\vartheta\sin\varphi + O(\frac{1}{\bar{r}}) = \sin\bar{\vartheta}\sin\varphi + O(\frac{1}{\bar{r}})$$

$$\langle \vec{n}, \partial_{\bar{z}}\rangle = \frac{\bar{z}}{\bar{r}} + O(\frac{1}{\bar{r}}) = \mathrm{sech}\psi\frac{R}{\bar{r}}\cos\vartheta + O(\frac{1}{\bar{r}}) = \cos\bar{\vartheta} + O(\frac{1}{\bar{r}}).$$

For the orthonormal frame field $\{e_1 = \frac{\partial_\vartheta}{R\beta^2 D^{\frac{1}{2}}}, e_2 = \frac{\partial_\varphi}{R\beta^2\sin\vartheta}, \vec{n} = \frac{D\bar{r}^2}{|D\bar{r}^2|}\}$ we have

$$\langle e_1, \partial_{\bar{x}}\rangle = \beta^4\langle e_1, D\bar{x}\rangle = \beta^2\frac{\bar{r}}{R}\Lambda^{-\frac{1}{2}}\partial_\vartheta(\frac{\bar{x}}{\bar{r}}) = \beta^2\frac{\bar{r}}{R}\Lambda^{-\frac{1}{2}}\partial_\vartheta(\frac{R}{\bar{r}}\sin\vartheta)\cos\varphi$$

$$= \mathrm{sech}\psi\beta^2\frac{R}{\bar{r}}(\frac{\lambda}{\Lambda})^{\frac{1}{2}}\cos\vartheta\cos\varphi = \cos\bar{\vartheta}\cos\varphi + O(\frac{1}{\bar{r}})$$

$$\langle e_2, \partial_{\bar{x}} \rangle = \beta^2 \sin^{-1}\vartheta(-\sin\vartheta\sin\varphi) = -\beta^2\sin\varphi = -\sin\varphi + O(\frac{1}{\bar{r}}),$$

$$\langle e_1, \partial_{\bar{y}} \rangle = \beta^4 \langle e_1, D\bar{y} \rangle = \beta^2 \frac{\bar{r}}{R}\Lambda^{-\frac{1}{2}}\partial_\vartheta(\frac{\bar{y}}{\bar{r}}) = \beta^2\frac{\bar{r}}{R}\Lambda^{-\frac{1}{2}}\partial_\vartheta(\frac{R}{\bar{r}}\sin\vartheta)\sin\varphi$$

$$= \text{sech}\psi\beta^2\frac{R}{\bar{r}}(\frac{\lambda}{\Lambda})^{\frac{1}{2}}\cos\vartheta\sin\varphi = \cos\bar{\vartheta}\sin\varphi + O(\frac{1}{\bar{r}})$$

$$\langle e_2, \partial_{\bar{y}} \rangle = \beta^2\sin^{-1}\vartheta(\sin\vartheta\cos\varphi) = \beta^2\cos\varphi = \cos\varphi + O(\frac{1}{\bar{r}})$$

and finally

$$\langle e_1, \partial_{\bar{z}} \rangle = \beta^2\frac{\bar{r}}{R}\Lambda^{-\frac{1}{2}}\{\cosh^2\psi - (\frac{\alpha}{\beta^3})^2\sinh^2\psi\}\partial_\vartheta(\frac{\bar{z}}{\bar{r}})$$

$$= \beta^2\frac{\bar{r}}{R}\Lambda^{-\frac{1}{2}}\{\cosh^2\psi - (\frac{\alpha}{\beta^3})^2\sinh^2\psi\}\partial_\vartheta(\text{sech}\psi\frac{R}{\bar{r}}\cos\vartheta)$$

$$= -\beta^2\frac{R}{\bar{r}}(\frac{\lambda}{\Lambda})^{\frac{1}{2}}\{\cosh^2\psi - (\frac{\alpha}{\beta^3})^2\sinh^2\psi\}\sin\vartheta$$

$$= -\sin\bar{\vartheta} + O(\frac{1}{\bar{r}})$$

$$\langle e_2, \partial_{\bar{z}} \rangle = 0.$$

Now given any ambient vector field X restricted to $\mathbb{S}^2_{\bar{r}}$,

$$K(X,\vec{n}) - K\langle X,\vec{n}\rangle = \langle X,\vec{n}\rangle(K(\vec{n},\vec{n}) - K) + \sum_i \langle X, e_i\rangle K(e_i,\vec{n})$$

$$= \langle X,\vec{n}\rangle\langle\vec{H},\vec{N}\rangle - \sum_i\langle X,e_i\rangle\alpha_{\vec{n}}(e_i)$$

$$= \langle X,\vec{n}\rangle\langle\vec{H},\vec{N}\rangle - \langle X,e_1\rangle\alpha_{\vec{n}}(e_1)$$

where we used $\sum_i \langle \text{II}_{\bar{r}}(e_i,e_i),\vec{N}\rangle = -K + K(\vec{n},\vec{n})$ in the second line and $\alpha_{\vec{n}} \propto d\vartheta$ in the third line (following from Proposition A.3.1 given that ϕ_n is independent of φ). From our calculations so far we recognize that $K(\partial_{\bar{x}},\vec{n}) - K\langle\partial_{\bar{x}},\vec{n}\rangle = F_1(\sin\vartheta)\cos\varphi$ and $K(\partial_{\bar{y}},\vec{n}) - K\langle\partial_{\bar{y}},\vec{n}\rangle = F_2(\sin\vartheta)\sin\varphi$ for some F_1 and F_2. So given that $\sqrt{\det g_{\mathbb{S}^2_{\bar{r}}}}$ is independent of φ we have $P_1 = P_2 = 0$. For P_3 we see

$$\langle\partial_{\bar z},\vec n\rangle\langle\vec H,\vec N\rangle-\langle e_1,\partial_{\bar z}\rangle\alpha_{\vec n}(e_1)$$

$$=[2M\frac{t}{\bar r}\lambda^{-\frac12}\{1-\lambda^{-1}+(\frac{\bar r}{R})^2\}\cos\bar\vartheta-2M\frac{R}{\bar r}\lambda^{-\frac12}\sin\bar\vartheta\frac{\bar r}{R}\lambda^{-\frac12}\eta_\vartheta]\frac{1}{\bar r^2}+O(\frac{1}{\bar r^3})$$

$$=2M\frac{\sinh\psi}{\cosh^2\psi}(\frac{R}{\bar r})^2\lambda^{-\frac12}[\cos^2\vartheta\{1-\sinh^2\psi(1-\tanh^2\psi\cos^2\vartheta)\}$$

$$+\cosh^2\psi\sin^2\vartheta]\frac{1}{\bar r^2}+O(\frac{1}{\bar r^3})$$

$$=2M\frac{\sinh\psi}{\cosh^2\psi}(\frac{R}{\bar r})^2\lambda^{-\frac12}[\cosh^2\psi-2\sinh^2\psi\cos^2\vartheta+\sinh^2\psi\tanh^2\psi\cos^4\vartheta]\frac{1}{\bar r^2}$$

$$+O(\frac{1}{\bar r^3})$$

$$=2M\sinh\psi(\frac{\bar r}{R})^2\lambda^{-\frac12}\frac{1}{\bar r^2}+O(\frac{1}{\bar r^3})$$

giving

$$\frac{1}{8\pi}\int_{\mathbb{S}^2_{\bar r}}\langle\partial_{\bar z},\vec n\rangle\langle\vec H,\vec N\rangle-\langle e_1,\partial_{\bar z}\rangle\alpha_{\vec n}(e_1)dA$$

$$=\frac{M}{2}\sinh\psi\int_0^\pi(\frac{\bar r}{R})^2\lambda^{-\frac12}(\frac{R}{\bar r})^2\Lambda^{\frac12}\sin\vartheta d\vartheta+O(\frac{1}{\bar r})$$

$$=\frac{M}{2}\sinh\psi\int_0^\pi(\frac{\Lambda}{\lambda})^{\frac12}\sin\vartheta d\vartheta+O(\frac{1}{\bar r})$$

$$=M\sinh\psi+O(\frac{1}{\bar r})$$

the result follows. □

Remark A.3.2. *(see [31]) In Proposition A.3.2 we recall for $X=\partial_{\bar z}$ that $\langle\partial_{\bar z},\vec n\rangle=\cos\bar\vartheta+O(\frac{1}{\bar r})$ giving for large $\bar r$, $\langle\partial_{\bar z},e_1\rangle e_1\approx-\sin\bar\vartheta\frac{\partial_{\bar\vartheta}}{\bar r}\approx\frac{1}{\bar r}\bar\nabla\cos\bar\vartheta$. So for any $\alpha_\star=\alpha_{\vec n}-d\phi_\star$ where $\phi_\star=\phi^0_\star+O(\frac{1}{\bar r})$ and $\phi^0_\star\in\mathcal{F}(\mathbb{S}^2)$*

$$\lim_{\bar{r}\to\infty}\int_{\mathbb{S}^2_{\bar{r}}}\langle\partial_{\bar{z}},\vec{n}\rangle\langle\vec{H},\vec{N}\rangle-\alpha_{\vec{n}}(\langle\partial_{\bar{z}},e_1\rangle e_1)dA$$

$$\begin{aligned}
&=\int_{\bar{\mathbb{S}}^2}\cos\bar{\vartheta}\lim(\bar{r}^2\langle\vec{H},\vec{N}\rangle)-\lim(\bar{r}\alpha_{\vec{n}})(\bar{\nabla}\cos\bar{\vartheta})d\sigma\\
&=\int_{\bar{\mathbb{S}}^2}\cos\bar{\vartheta}\lim(\bar{r}^2\langle\vec{H},\vec{N}\rangle)+\bar{\nabla}\cdot(\lim(\bar{r}\alpha_{\vec{n}}))\cos\bar{\vartheta}d\sigma\\
&=\int_{\bar{\mathbb{S}}^2}\cos\bar{\vartheta}(\lim(\bar{r}^2\langle\vec{H},\vec{N}\rangle)+\lim(\bar{r}^3\nabla_{\mathbb{S}^2_{\bar{r}}}\cdot\alpha_{\vec{n}}))d\sigma\\
&=\int_{\bar{\mathbb{S}}^2}\cos\bar{\vartheta}(\lim(\bar{r}^2\langle\vec{H},\vec{N}\rangle)+\lim(\bar{r}^3\nabla_{\mathbb{S}^2_{\bar{r}}}\cdot\alpha_\star)+\lim(\bar{r}^2\Delta_{\mathbb{S}^2_{\bar{r}}}\bar{r}\phi_\star))d\sigma\\
&=\int_{\bar{\mathbb{S}}^2}\cos\bar{\vartheta}(\lim(\bar{r}^2\langle\vec{H},\vec{N}\rangle)+\lim(\bar{r}^3\nabla\cdot\alpha_\star)-2\lim(\bar{r}\phi_\star))d\sigma
\end{aligned}$$

having used $\bar{\Delta}_{\mathbb{S}^2}\cos\bar{\vartheta}=-2\cos\bar{\vartheta}$ *after an integration by parts twice to obtain the final term of the final line.*

From this remark we see that the momentum (at least in the limit) is completely encoded within the connection 1-form provided we choose an orthonormal frame that yields $\lim_{\bar{r}\to\infty}\bar{r}\varphi_\star=\lim_{\bar{r}\to\infty}\frac{\bar{r}^2}{2}\langle\vec{H},\vec{N}\rangle$. Noticing that $\vec{H}=\langle\vec{H},\nu\rangle\nu-\langle\vec{H},\nu^\star\rangle\nu^\star$ is dominated by $\langle\vec{H},\nu\rangle\nu\approx-\frac{2}{\bar{r}}\nu$ for large $\bar{r}$ we may choose the orthonormal frame field $\{\nu_H,\nu_H^\star\}$ whereby,

$$\begin{aligned}
\nu_H&:=-\frac{\vec{H}}{H}=\cosh\varphi_H\vec{n}+\sinh\varphi_H\vec{N}\\
\nu_H^\star&:=\sinh\varphi_H\vec{n}+\cosh\varphi_H\vec{N}
\end{aligned}$$

with associated connection 1-form given by $\alpha_H=\alpha_n-d\varphi_H$.

Bibliography

[1] S. Alexakis. The penrose inequality on perturbations of the schwarzschild exterior. *arXiv preprint arXiv:1506.06400*, 2015.

[2] R. Arnowitt, S. Deser, and C. W. Misner. Republication of: The dynamics of general relativity. *General Relativity and Gravitation*, 40(9):1997–2027, 2008.

[3] R. Bartnik. Phase space for the einstein equations. *Communications in Analysis and Geometry*, 13(5):845–885, 2005.

[4] G. Bergqvist. On the penrose inequality and the role of auxiliary spinor fields. *Classical and Quantum Gravity*, 14(9):2577, 1997.

[5] H. Bray, S. Hayward, M. Mars, and W. Simon. Generalized inverse mean curvature flows in spacetime. *Communications in mathematical physics*, 272(1):119–138, 2007.

[6] H. L. Bray. Proof of the riemannian penrose inequality using the positive mass theorem. *Journal of Differential Geometry*, 59(2):177–267, 2001.

[7] H. L. Bray, J. L. Jauregui, and M. Mars. Time flat surfaces and the monotonicity of the spacetime hawking mass ii. *Ann. Henri Poincare*, 17(6):1457–1475, 2016.

[8] H. L. Bray and D. A. Lee. On the riemannian penrose inequality in dimensions less than eight. *Duke Mathematical Journal*, 148(1):81–106, 2009.

[9] D. Christodoulou. *Mathematical problems of general relativity I.* European Mathematical Society, 2008.

[10] R. Geroch. Energy extraction. *Ann. New York Acad. Sci*, 224:108–17, 1973.

[11] E. Gourgoulhon and J. L. Jaramillo. A 3+ 1 perspective on null hypersurfaces and isolated horizons. *Physics Reports*, 423(4):159–294, 2006.

[12] S. W. Hawking. Gravitational radiation in an expanding universe. *Journal of Mathematical Physics*, 9(4):598–604, 1968.

[13] S. W. Hawking and R. Penrose. The singularities of gravitational collapse and cosmology. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 314, pages 529–548. The Royal Society, 1970.

[14] S. A. Hayward. Gravitational energy in spherical symmetry. *Physical Review D*, 53(4):1938, 1996.

[15] G. Huisken and T. Ilmanen. The inverse mean curvature flow and the riemannian penrose inequality. *Journal of Differential Geometry*, 59(3):353–437, 2001.

[16] P. S. Jang and R. M. Wald. The positive energy conjecture and the cosmic censor hypothesis. *Journal of Mathematical Physics*, 18(1):41–44, 1977.

[17] M. Ludvigsen and J. Vickers. An inequality relating total mass and the area of a trapped surface in general relativity. *Journal of Physics A: Mathematical and General*, 16(14):3349, 1983.

[18] M. Mars and A. Soria. The asymptotic behaviour of the hawking energy along null asymptotically flat hypersurfaces. *Classical and Quantum Gravity*, 32(18):185020, 2015.

[19] M. Mars and A. Soria. On the penrose inequality along null hypersurfaces. *Classical and Quantum Gravity*, 33(11):115019, 2016.

[20] W. H. Meeks and S.-T. Yau. Topology of three dimensional manifolds and the embedding problems in minimal surface theory. *Annals of Mathematics*, 112(3):441–484, 1980.

[21] B. O'Neill. *Semi-Riemannian geometry with applications to relativity*, volume 103. Academic press, 1983.

[22] A. R. Parry. A survey of spherically symmetric spacetimes. *Analysis and Mathematical Physics*, 4(4):333–375, 2014.

[23] R. Penrose. Gravitational collapse: The role of general relativity. Technical report, Birkbeck Coll., London, 1969.

[24] T. Regge and C. Teitelboim. Role of surface integrals in the hamiltonian formulation of general relativity. *Annals of Physics*, 88(1):286–318, 1974.

[25] J. Sauter. *Foliations of null hypersurfaces and the Penrose inequality*. PhD thesis, ETH Zurich, 2008.

[26] R. Schoen and S.-T. Yau. On the proof of the positive mass conjecture in general relativity. *Communications in Mathematical Physics*, 65(1):45–76, 1979.

[27] R. Schoen and S.-T. Yau. Proof of the positive mass theorem. ii. *Communications in Mathematical Physics*, 79(2):231–260, 1981.

[28] R. Schoen and S. T. Yau. Proof that the bondi mass is positive. *Physical Review Letters*, 48(6):369, 1982.

[29] R. M. Wald. *General relativity*. University of Chicago press, 2010.

[30] M.-T. Wang, Y.-K. Wang, and X. Zhang. Minkowski formulae and alexandrov theorems in spacetime. *Journal of Differential Geometry*, 105(2):249–290, 2017.

[31] M.-T. Wang and S.-T. Yau. Limit of quasilocal mass at spatial infinity. *Communications in Mathematical Physics*, 296(1):271–283, 2010.

[32] E. Witten. A new proof of the positive energy theorem. *Communications in Mathematical Physics*, 80(3):381–402, 1981.

Biography

Henri Petrus Roesch was born in Pretoria, South Africa on May 12, 1989 to parents Elize and Henning Roesch. Most recently, Roesch recieved Ph.D and M.A. degrees in mathematics from Duke University in Durham, North Carolina. Prior to that, he recieved a B.Sc. degree in mathematics with theoretical physics from University College London and a M.A.St degree in theoretical physics from Cambridge University in the United Kingdom.

In July 2017, Roesch will be joining the faculty of the University of California, Irvine Department of Mathematics as a National Science Foundation Postdoctoral Fellow. In July 2018, Roesch will be joining the faculty of Columbia University Department of Mathematics as a J.F. Ritt Assistant Professor.