The Hyperbolic Positive Mass Theorem and Volume Comparison Involving Scalar Curvature

by

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Dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
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ABSTRACT

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Abstract

In this thesis, we study how the scalar curvature relates to the ADM mass and volume of the manifold.

The positive mass theorem, first proven by Schoen and Yau in 1979, states that nonnegative local energy density implies nonnegative total mass. The harmonic level set technique pioneered by D. Stern [Ste19] has been used to prove a series of positive mass theorems, such as the Riemannian case [BKKS19], the spacetime case [HKK20] and the charged case. Using this novel technique, we prove the hyperbolic positive mass theorem in the spacetime setting, as well as some rigidity cases. A new interpretation of mass is introduced in this context. Then we solve the spacetime harmonic equation in the hyperbolic setting. We not only prove the positive mass theorem, but we also give a lower bound for the total mass without assuming the nonnegativity of the local energy density.

Additionally, we prove a scalar curvature volume comparison theorem, assuming some boundedness for Ricci curvature. The proof relies on the perturbation of the scalar curvature [BM11].

Acknowledgements

First of all, I would like to express my thanks to my advisor, Hugh Bray, for his support, enthusiasm and patience. I greatly enjoyed the past five years as a graduate student.

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Chapter 1

Introduction

General relativity is a theory of gravity. The central equation in general relativity is Einstein's equation,

$$G = 8\pi T, \tag{1.1}$$

where $G = \mathrm{Ric} - \frac{1}{2}Sg$ is the Einstein curvature tensor, and T is the stress-energy tensor. The Einstein equation can be identified as the Euler-Langangian equation of the Einstein-Hilbert action, which is the integral of scalar curvature plus some matter field terms. The fundamental rule in general relativity that none can travel faster than light implies $T(v, w) \geq 0$ for any future pointing non-spacelike vectors v, w. Suppose we select a three-dimension spacelike slice (M, g, k), where g is the metric and k is the second fundamental form. Let $\mathbf{n}$ be the normal vector of the slice in the spacetime. Then $T(v, w) \geq 0$ implies the energy density $\mu = G(\mathbf{n}, \mathbf{n})$ and momentum density $J = G(\mathbf{n}, \cdot)$ satisfy the dominant energy condition (DEC): $\mu \geq |\boldsymbol{J}|$.

The positive mass theorem (PMT) states that the total mass is nonnegative. The Riemannian PMT, $k = 0$ case, was first proven by Schoen and Yau [SY79]. It asserts that for an asymptotically flat manifold with nonnegative scalar curvature, the Arnowitt-Deser-Misner (ADM) mass is nonnegative. Soon after, Schoen and Yau reduced the spacetime PMT to $k = 0$ by solving the Jang equation. Another major approach for the positive mass theorems was developed by Witten [Wit81]. Witten applied spinor techniques to give another proof of the positive mass theorem for spin manifolds. Later, Huisken and Illmanen used inverse mean curvature to give a new proof of the Riemannian PMT. They established the Riemannian Penrose inequality for one black hole which was a stronger inequality relating the mass and the area of black holes. The Riemannian Penrose inequality for multiple black holes was proven by Bray using a new conformal flow [Bra01]. There is also a Ricci flow approach for the Riemannian PMT [Li18].

The positive mass theorem is closely related to the non-existence of a metric with positive scalar curvature on T^3 ([Loh99]), which is called the Geroch conjecture. It was obtained by Schoen and Yau [SY78, SY79] in dimensions less than 8 and later proven by Gromov and Lawson [GLJ80] for spin manifolds in all dimensions . Recently, Stern [Ste19] observed that the level set of harmonic functions can be used to solve this problem. This approach is also used to prove the Riemannian PMT, the spacetime PMT, the PMT with the charge and the hyperbolic PMT. In dimension three, there are important advantages to this new approaches, as well as new results.

The first part of the thesis uses the harmonic level set technique [Ste19] to prove the hyperbolic positive mass theorem in the spacetime setting. It is a joint work with H. Bray, S. Hirsch, D. Kazaras and M. Khuri.

The second part of the thesis is on scalar volume comparison. Scalar curvature appears in the expansion of the volume of a geodesic ball. Therefore, locally, larger scalar curvature implies a smaller volume for the geodesic ball. However, it is an infinitesimal property and there is no similar global results. Meanwhile, unlike scalar curvature, a larger Ricci curvature implies a smaller volume globally, according to Bishop theorem. If we add a lower bound on Ricci curvature, Bray's football theorem is a sharp volume comparison theorem involving scalar curvature in dimension 3. In the thesis, we obtain similar results in higher dimensions, assuming a upper bound on Ricci curvature or the manifold is axisymmetric.

Chapter 2

Asymptotically hyperbolic manifolds

Asymptotically hyperbolic manifolds arise naturally in two circumstances. One is the asymptotically totally geodesic spacelike hypersurfaces in asymptotically Ads (Anti-de Sitter) spacetime; another is the asymptotically umbilic slice in Minkowski spacetime, i.e., $k \approx g$. The positive mass theorem in AdS spacetime setting is investigated in [XZ08, Zha04, WX12] using spinors. We focus on the second type, the model space of which is the hyperboloidal hypersurface $t = \sqrt{1+r^2}$ in Minkowski spacetime.

Min-Oo proved a scalar curvature rigidity theorem for strongly asymptotically hyperbolic manifolds [MO89]. Inspired by that, Wang [Wan01] defined the mass for asymptotically hyperbolic manifolds (M^n, g), $n \geq 3$, with the expansion at infinity,

$$g = \frac{1}{(\sinh \rho)^2}(d\rho^2 + r^2 d\sigma^2 + \frac{\rho^n}{n} m + \mathcal{O}(\rho^{n+1})), \tag{2.1}$$

where $d\sigma^2$ is the round metric on S^{n-1}, and m is a symmetric 2-tensor on S^2, called the mass aspect tensor. The total energy p_0 and total momentum vector (p_1, p_2, p_3) are defined as below

$$p_0 = \frac{1}{16\pi}\int_{S^2} \mathrm{tr}_{d\sigma^2} m dA, \quad p_i = \frac{1}{16\pi}\int_{S^2} x^i \, \mathrm{tr}_{d\sigma^2} m dA, \tag{2.2}$$

where $(x^1, x^2, x^3) \in S^2 \subset \mathbb{R}^3$, i.e., $|(x^1, x^2, x^3)| = 1$. If the scalar curvature of (M^n, g) satisfies $R \geq -6$, Wang used spinors to prove the positive mass theorem

$p_0 \geq \sqrt{p_1^2 + p_2^2 + p_3^2}$. Therefore, we can define the mass

$$\mathfrak{m} = p_0 - \sqrt{p_1^2 + p_2^2 + p_3^2}. \tag{2.3}$$

Also using spinors, Chruściel and Herzlich [CH03] proved the hyperbolic positive mass theorem with significantly more general asymptotics than (2.1). See Section 2.1 for a detailed discussion of the asymptotic behavior of g. There are also several non-spin proofs available in literature. In the spirit of the Schoen-Yau minimal surface method, Anderson, Cai and Galloway [ACG08] employed the brane action to prove the hyperbolic positive mass theorem in dimensions 3-7 under the additional assumption that the mass-aspect function $\operatorname{tr}_{d\sigma^2} m$ does not change sign. Moreover, Chruściel and Delay [CD19] have reduced the hyperbolic positive mass theorem in dimensions 3-7 to the spacetime positive mass theorem [SY81b, Wit81, Eic13, EHLS11, HKK20]. Finally, there was the recent paper [Sak20] by Sakovich where the spacetime hyperbolic positive mass theorem is proven in dimension 3. Lastly, we point out the paper [HL19] where Huang, Jang and Martin settle the rigidity case of the hyperbolic positive mass theorem in the umbilic case with full generality.

To this list we add a new proof based on the harmonic level set method and an interpolation method. The level set method pioneered in [Ste19] has been successfully used in proving the Riemannian and the spacetime positive mass theorems [BKKS19, HKK20]. Also using spacetime harmonic functions but otherwise a very different method compare to [HKK20], we give a new proof of the hyperbolic positive mass theorem and obtain the following lower bound on the mass.

Theorem 1. *Let (M, g, k) be a three dimensional, complete, simply connected asymptotically hyperbolic manifold. Let E and (P_1, P_2, P_3) be the total energy and total* *momentum defined in subsection 2.1.3. Then*

$$E - P_i \geq \frac{1}{16\pi} \int_M \left[\frac{|\nabla^2 u + k|\nabla u||^2}{|\nabla u|} + 2(\mu|\nabla u| + \langle J, \nabla u \rangle) \right] dV \tag{2.4}$$

where u *satisfies* $\Delta u + |\nabla u| \mathrm{Tr}_g k = 0$ *and is asymptotic to a Minkowski null coordinate function* $-t - x_i$ *at* ∞.

We claim that the hyperbolic positive mass theorem is implied by our result. Suppose $|P| \neq 0$, then we can choose u to be asymptotic to $-t - \sum_i |P|^{-1} x_i P_i$. Equation (2.4) becomes

$$E - |P| \geq \frac{1}{16\pi} \int_M \left[\frac{|\nabla^2 u + k|\nabla u||^2}{|\nabla u|} + 2(\mu|\nabla u| + \langle J, \nabla u \rangle) \right] dV. \tag{2.5}$$

Hence we do indeed obtain the hyperbolic positive mass theorem as a corollary.

Here is the organization of this chapter. We introduce the background and proof strategy in Sections 2.1 and 2.2. In Section 2.3, we solve the spacetime harmonic equation in the asymptotically hyperbolic setting. A more delicate expansion of the solution is given in Section 2.4. We establish the main inequality in Sections 2.5 and 2.6. Finally, in Section 2.7, we prove two rigidity results.

Throughout this chapter, we use the letter C to denote constants, and $C_{*,*}$ is the constant depending only on the quantities appearing in the subscript.

2.1 Setup and Definitions

2.1.1 Hyperbolic space

There are many ways to represent hyperbolic space $(\mathbb{H}^3, b)$ such as the Poincaré disc model and the Beltrami-Klein model. It turns out that for our proofs and computations there are three models which are the most convenient to use.

The hyperboloidal model

We denote with $\mathbb{R}^{3,1}$ the Minkowski spacetime with spatial coordinates x, y, z and time coordinate t. Hyperbolic space $\mathbb{H}^3$ can be identified as the unit sphere of Minkowski spacetime $\mathbb{M}^4$, i.e. $\mathbb{H}^3 = \{(x, y, z, t) : t^2 = 1+r^2\}$ where $r = \sqrt{x^2 + y^2 + z^2}$. Moreover, we may write the hyperbolic metric b as

$$b = \frac{1}{1 + r^2} dr^2 + r^2 d\sigma^2, \tag{2.6}$$

where $d\sigma^2$ is the round metric on the unit sphere.

Polar Coordinate

With $\mathbb{H}^3 \subset \mathbb{M}^4$, let T be the geodesic distance from $(0, 0, 0, 1)$. We have the metric

$$b = dT^2 + \sinh^2 T d\sigma^2. \tag{2.7}$$

The r in the hyperboloidal model can be written as $r = \sinh T$.

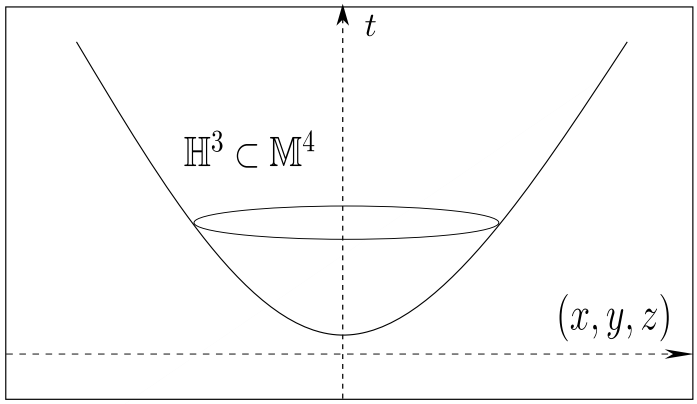

Figure 2.1: Hyperboloidal Model

Poincaré halfspace model

We denote with x_1, x_2, x_3 the Cartesian coordinates in $\mathbb{R}^3$ and set $\mathbb{R}^3_+ := \{x_3 > 0\}$. We can identify $\mathbb{H}^3$ as $\mathbb{R}^3_+$ with the metric

$$b = \frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}. \tag{2.8}$$

We can transform the coordinates from the hyperboloidal model to the coordinates of the Poincaré halfspace model via the transformation formula

$$(x, y, z, t) \to \left(\frac{2y}{t+x}, \frac{2z}{t+x}, \frac{2}{t+x} \right). \tag{2.9}$$

2.1.2 Weighted Hölder spaces on hyperbolic space

As in the asymptotically flat setting we need to introduce function spaces which take the decay of functions and tensors into account.

Definition 2. The weighted Hölder space $C^{k,\alpha}_\rho(\mathbb{H}^3)$ is the space of $C^{k,\alpha}_{loc}$ functions whose weighted Hölder norm defined by

$$\|f\|_{C^{k,\alpha}_\rho} = \sum_{0\le l\le k} \|t^\rho \nabla^l f\|_b + \|t^\rho \nabla^k f\|_{b,\alpha} \tag{2.10}$$

is finite. Here $t = \sqrt{1+r^2} = \sqrt{1+x^2+y^2+z^2}$ is the hyperboloidal time coordinate.

We use $O_{k,\alpha}(r^{-\rho})$ to denote functions or tensors that belongs to $C^{k,\alpha}_\rho(M)$.

Observe that this definition does not impose higher order decay on the derivatives as the asymptotically flat setting. This is natural in the hyperbolic setting. For instance, we have $|\nabla(x+t)|_b = x+t$. Moreover, this definition extends canonically to $C^{k,\alpha}_\rho(M)$ for asymptotically hyperbolic manifolds (M,g), which we introduce in the next subsection. For more details see [HJM19].

2.1.3 Asymptotically hyperbolic manifolds

Although all our results hold true for very general asymptotics, we also include the corresponding statements for Wang's asymptotics throughout the paper for the reader's convenience. We first give a concrete definition of the total energy E and total momentum vector P under Wang's asymptotics. Then we introduce E and P under the general asymptotics defined in [CH03, CJL04].

Wang's asymptotics

Definition 3. We say that a manifold (M,g,k) is asymptotically hyperbolic with Wang's asymptotics if there exists a chart into $\mathbb{H}^3$ (equipped with hyperboloidal coordinates) outside a large compact set such that the metric in this chart has the form

$$g = \frac{1}{1+r^2}dr^2 + r^2\left(d\sigma^2 + \frac{m}{r^3} + O_2(r^{-4})\right), \tag{2.11}$$

$$k = \frac{1}{1+r^2}dr^2 + r^2\left(d\sigma^2 + \frac{p}{r^3} + O_1(r^{-4})\right). \tag{2.12}$$

We define the energy E and the components P_i of the momentum vector by

$$E = \frac{1}{16\pi}\int_{S^2} \mathrm{Tr}_{d\sigma^2}(m+2p)dA, \quad P_i = \frac{1}{16\pi}\int_{S^2} x^i\,\mathrm{Tr}_{d\sigma^2}(m+2p)dA, \tag{2.13}$$

where $(x^1, x^2, x^3) \in S^2 \subset \mathbb{R}^3$, i.e. $|(x^1, x^2, x^3)| = 1$.

The mass is given by

$$\mathfrak{m} = E - \sqrt{P_1^2 + P_2^2 + P_3^2}. \tag{2.14}$$

General Asymptotics

Definition 4. Let $\tau \in (\frac{3}{2}, 3)$. We say that an initial data set (M, g, k) is asymptotically hyperbolic of order τ, if outside a compact set C there exists a chart into $\mathbb{H}^3$ such that the metric g and the symmetric 2-tensor k satisfy

$$(g-b) \in C^{2,\alpha}_\tau(M \setminus C), \qquad (g-k) \in C^{1,\alpha}_\tau(M \setminus C) \tag{2.15}$$

and

$$|\mu| + |J| \in C^{0,\alpha}_{-3-\epsilon}(M) \tag{2.16}$$

for some $\epsilon > 0$. Here μ and J are the energy- and momentum density given by

$$\mu = \frac{1}{2}\left(R_g + K^2 - |k|_g^2\right), \qquad J = \mathrm{div}_g\,(k - Kg) \tag{2.17}$$

where $K = \mathrm{Tr}_g k$.

Motivated by [HJM19, CJL04], we define the functional

$$H(V) = \lim_{r\to\infty}\int_{S_r}\left[V\left(\mathrm{div}_b e - d(\mathrm{Tr}_b e)\right) + (\mathrm{Tr}_b(e+2f))dV - (e+2f)(\nabla_b V,\cdot)\right](\nu_\rho)d\mu_b \tag{2.18}$$

where $\nu_\rho = \sqrt{1+r^2}\partial_\rho$ is the outward unit normal vector on $S_r \subset \mathbb{H}^3$, $e = g - b$, and $f = k - g$. We explain in subsection 2.5 how this definition relates to the original one from [CJL04]. Then, the components of the energy-momentum vector are given by

$$E = \frac{1}{16\pi}H(t), \quad P_1 = \frac{1}{16\pi}H(x), \quad P_2 = \frac{1}{16\pi}H(y), \quad P_3 = \frac{1}{16\pi}H(z) \tag{2.19}$$

where again $x, y, z, t = \sqrt{1+x^2+y^2+z^2}$ are the hyperboloidal coordinates. Again, we set the mass to be

$$\mathfrak{m} = E - \sqrt{P_1^2 + P_2^2 + P_3^2}. \tag{2.20}$$

The mass $\mathfrak{m}$ is a geometric invariant as shown in [Wan01] and [CJL04].

2.2 Proof strategy

Our proof consists of two ingredients: spacetime harmonic functions which were introduced in [HKK20] and a novel interpolation method. We motivate both concepts in this section.

2.2.1 Spacetime harmonic functions

Spacetime harmonic functions were introduced in [HKK20] in order to give a new proof of the spacetime positive mass theorem. In order to make the current paper self contained we recall the main aspects.

Review of the Riemannian setting

Up to the year 2019 there have been two major tools to study scalar curvature: minimal surfaces and spinors. In [Ste19], Stern introduced a new method to study scalar curvature which is based on harmonic maps and a combination of Bochner's identity, Gauss-Codazzi equations and Gauss-Bonnet's theorem. In [BKKS19], these techniques have been extended to give a new proof of the Riemannian positive mass theorem. In particular, they proved the mass formula

$$\mathfrak{m} \geq \frac{1}{16\pi} \int_{M_{ext}} \left[\frac{|\nabla^2 u|^2}{|\nabla u|} + R|\nabla u| \right]. \tag{2.21}$$

Here, R is the scalar curvature of the asymptotically flat manifold (M, g), M_{ext} is the exterior region of M and u is a harmonic function asymptotic to the coordinate function x. This resembles the mass formula proven by Witten [Wit81] using spinors, but it also holds true for manifolds which do not possess non-negative scalar curvature.

Motivation of null spacetime harmonic functions

As shown in [Ste19] and [BKKS19], the Laplace equation presents a good PDE to study the scalar curvature of asymptotically flat manifolds. This raises the question what the most natural PDE for initial data sets is. To answer this question, we assume that (M, g, k) embeds into a spacetime $(\hat{M}^4, \hat{g})$. Then, we can solve the hypersurface spacetime Laplace equation $g^{ij}\hat{\nabla}^2_{ij}u = 0$ in M, i.e.,

$$0 = \hat{\Delta}\hat{u} = g^{ij}\left(\nabla_{ij}\hat{u} - k_{ij}\mathbf{n}(\hat{u})\right) = \Delta\hat{u} - (\mathrm{Tr}_g k)\,\mathbf{n}(\hat{u}) \quad \text{on} \quad M, \tag{2.22}$$

where $\mathbf{n}$ is the unit timelike normal to the slice. This equation, however, does not depend solely on the restriction $u = \hat{u}|_M$ due to the presence of the normal derivative. A choice for $\mathbf{n}(\hat{u})$ must be made in order to obtain a purely intrinsic equation on the slice. It turns out that the desired choice for our purposes is to choose the normal derivative so that the spacetime gradient of $\hat{u}$ is null, that is, $\mathbf{n}(\hat{u}) = -|\nabla u|$. With this in mind, we make the following definitions. Given an asymptotically hyperbolic manifold (M, g, k), we set

$$\tilde{\nabla}^2 u = \nabla^2 u + k|\nabla u|. \tag{2.23}$$

A function $u \in C^2(M)$ is *(null) spacetime harmonic* if $\tilde{\Delta}u = 0$ where

$$\tilde{\Delta}u = \mathrm{Tr}_g\tilde{\nabla}^2 u = \Delta u + K|\nabla u|. \tag{2.24}$$

Here, we denote $K = \mathrm{Tr}_g(k)$.

Informal computation of spacetime integral formula

We show how spacetime harmonic functions are related to the geometry of initial data sets. This computation is already contained in [HKK20], though we include an informal discussion of this computation for the sake of completeness.

In the following we assume $|\nabla u|$ is not vanishing and u has no spherical level sets. By Bochner's identity, we have

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) - \langle \nabla u, \nabla(K|\nabla u|)\rangle. \tag{2.25}$$

Hence,

$$\Delta|\nabla u| = \frac{1}{|\nabla u|}(|\nabla^2 u|^2 - |\nabla|\nabla u||^2 + \mathrm{Ric}(\nabla u, \nabla u) - \langle \nabla u, \nabla(K|\nabla u|)\rangle). \tag{2.26}$$

Observe that on regular level sets of u, there exists a unit normal ν which satisfies $\nu = \frac{\nabla u}{|\nabla u|}$. Also, recall from [Ste19] the formulas

$$h_{ij} = \frac{1}{|\nabla u|}\nabla_{ij} u, \quad H = \frac{1}{|\nabla u|}(-K|\nabla u| - \nabla_{\nu\nu} u). \tag{2.27}$$

Therefore, we have on regular level sets

$$|\nabla u| H = -K|\nabla u| - \nabla_{\nu\nu} u \tag{2.28}$$

which leads to

$$|\nabla u|^2(H^2 - |A|^2) = 2|\nabla|\nabla u||^2 - |\nabla^2 u|^2 + (K|\nabla u|)^2 + 2K|\nabla u|\nabla_{\nu\nu} u. \tag{2.29}$$

The Gauß equations imply together with (2.26)

$$\Delta|\nabla u| = \frac{1}{2|\nabla u|}(|\nabla^2 u|^2 + |\nabla u|^2(R_M - R_\Sigma) - 2\langle \nabla u, \nabla(K|\nabla u|)\rangle) \tag{2.30}$$

$$+ \frac{1}{2}K^2|\nabla u| + K\nabla_{\nu\nu} u. \tag{2.31}$$

Next, we use the notation $\tilde{\nabla}_{ij} = \nabla_{ij} + k_{ij}|\nabla u|$ for the spacetime Hessian of u to obtain

$$\Delta|\nabla u| = \frac{1}{2|\nabla u|}\Big(|\tilde{\nabla}^2 u|^2 + |\nabla u|^2(R_M - R_\Sigma) + 2\langle \nabla u, \nabla(K|\nabla u|)\rangle + K^2|\nabla u|^2 \tag{2.32}$$

$$- 2K|\nabla u|\nabla_{\nu\nu}u\Big) + \frac{1}{2|\nabla u|}(-2\nabla_{ij}uk_{ij}|\nabla u| - |k|^2|\nabla u|^2). \tag{2.33}$$

Observe that

$$\langle \nabla u, \nabla|\nabla u|\rangle = |\nabla u|\nabla_{\nu\nu}u. \tag{2.34}$$

Hence, the above term simplifies to

$$\Delta|\nabla u| = \frac{1}{2|\nabla u|}\Big(|\tilde{\nabla}^2 u|^2 + |\nabla u|^2(R_M - R_\Sigma) + 2|\nabla u|\langle \nabla u, \nabla K\rangle + K^2|\nabla u|^2\Big) \tag{2.35}$$

$$+ \frac{1}{2|\nabla u|}(-2\nabla_{ij}uk_{ij}|\nabla u| - |k|^2|\nabla u|^2). \tag{2.36}$$

Integrating by parts yields

$$-\int_{M_\rho} \nabla_{ij}uk_{ij} = \int_{M_\rho} \nabla_i u\nabla_j k_{ij} - \int_{\partial M_\rho} \nabla_i uk_{i\nu} \tag{2.37}$$

where ν is the outer unit normal to ∂M_ρ and M_ρ is the part of M enclosed by a large coordinate sphere S_r. Combining this with the identities

$$2\mu = R + K^2 - |k|^2, \quad J_i = \operatorname{div}(k - Kg)_i, \tag{2.38}$$

yields the spacetime integral formula.

Proposition 5. *Suppose $|\nabla u|$ is not vanishing and u has no spherical level sets.*

Then we have

$$\int_{\partial M_\rho} (\nabla_\upsilon |\nabla u| + \nabla_i u k_{i\upsilon}) = \int_{M_\rho} \frac{|\tilde{\nabla}^2 u|^2}{|\nabla u|} + 2(\mu |\nabla u| + \langle J, \nabla u \rangle) - |\nabla u| R_\Sigma \tag{2.39}$$

This formula is also valid without the assumptions in the proposition, see Proposition 3.2 in [HKK20]. In [HKK20], it has been shown that the boundary term $\int_{\partial M_r} (\nabla_\upsilon |\nabla u| + \nabla_i u k_{i\upsilon}) + \int_{M_\rho} |\nabla u| R_\Sigma$ in the asymptotically flat setting does converge to the energy and momentum for $M_\rho \to M$. Our goal is to proceed similarly in the asymptotically hyperbolic setting. However, it turns out that one has to proceed in a very different fashion.

2.2.2 Interpolation

Next, we discuss our new interpolation method which gives a new interpretation of mass even in the asymptotically flat setting. More precisely, we can consider mass to be the amount of negative scalar curvature we have to 'pay' to deform our asymptotically flat manifold into Euclidean space. But first, we elaborate on why we cannot directly apply the proof in the asymptotically flat setting for the asymptotically hyperbolic manifolds.

Difficulties of a direct proof

In the asymptotically flat setting, we have much more control of the PDE $\Delta u = -K|\nabla u|$ than in the asymptotically hyperbolic setting. More precisely, in the asymptotically hyperbolic setting

- the mean curvature K is not decaying but is approaching 3 at infinity. Hence, we cannot regard the PDE as being 'essentially' linear.

- the term $|\nabla u|$ is not going to 1 at infinity. Instead, it blows up almost everywhere except for one direction where $|\nabla u|$ goes to 0. Hence, we are not in a rotationally symmetric setting which has to be taken into account for barrier constructions.

- we do not obtain better controls on the derivatives of u whereas we have $u - x = O_2(r^{1-q})$ in the asymptotically flat setting. This presents huge technical difficulties when computing the boundary terms at infinity - even after successfully addressing the previous two issues.

Therefore, it is desirable to find a shortcut for the computation at infinity. We first motivate with an example in the asymptotically flat, Riemannian setting.

Interpolation from Schwarzschild to Euclidean space

We begin with interpolating between a spatial Schwarzschild manifold of mass $\mathfrak{m}$ and Euclidean space, and we show that the whole mass is 'stored' in an integral over the scalar curvature. Interestingly, this is independent of the way we interpolate.

Consider the the Schwarzschild and the interpolated Schwarzschild metric

$$\tilde{g} = \left(1 + \frac{\mathfrak{m}}{2r}\right)^4 \delta, \quad g = \left(1 + \frac{m(r)}{2r}\right)^4 \delta \tag{2.40}$$

where $m(r) = \mathfrak{m}$ on the ball B_ρ, $\rho \gg 1$, and $m(r) = 0$ outside $B_{\rho+1}$. The scalar curvature of this metric is given by

$$R = -8\left(1 + \frac{m(r)}{2r}\right)^{-5} \Delta \left(1 + \frac{m(r)}{2r}\right) = (-8 + \mathcal{O}(r^{-1}))\Delta \left(1 + \frac{m(r)}{2r}\right). \tag{2.41}$$

Next, we obtain from a straightforward computation

$$\Delta\left(1+\frac{m(r)}{2r}\right)=\frac{m''(r)}{2r} \tag{2.42}$$

which implies

$$R=(-4r^{-1}+\mathcal{O}_1(r^{-2}))m''(r). \tag{2.43}$$

Recall the mass formula from equation (2.21) which states in the Schwarzschild manifold,

$$16\pi\mathfrak{m}\geq\int_M\left(\tilde{R}|\nabla u|+\frac{|\nabla^2 u|^2}{|\nabla u|}\right)\geq\int_{B_\rho}\left(R|\nabla u|+\frac{|\nabla^2 u|^2}{|\nabla u|}\right) \tag{2.44}$$

where u is a harmonic function asymptotic to a coordinate function. Moreover, the mass formula states that in the interpolated Schwarzschild manifold,

$$-\int_{A_{\rho+1,\rho}}\left(R|\nabla u|+\frac{|\nabla^2 u|^2}{|\nabla u|}\right)\geq\int_{B_\rho}\left(R|\nabla u|+\frac{|\nabla^2 u|^2}{|\nabla u|}\right), \tag{2.45}$$

where $A_{\rho,\rho+1}$ is the annulus $B_{\rho+1}\backslash B_\rho$. Since $|\nabla u|=1+\mathcal{O}(r^{-1})$ and $|S_r|=4\pi r^2+\mathcal{O}(r)$, we obtain by the divergence theorem

$$\int_{\rho-1}^{\rho+2}|S_r||\nabla u|R=16\pi\int_{\rho-1}^{\rho+2}(-r+\mathcal{O}_1(1))m''(r)\to-16\pi\mathfrak{m} \tag{2.46}$$

for $\rho\to\infty$. Hence, all the mass is 'stored' in the scalar curvature integral over an annulus near infinity. Instead of thinking of mass as being the measurement how far our asymptotically flat manifold differs from Euclidean space at infinity, we can interpret mass as the amount of negative scalar curvature we have to 'pay' to deform our asymptotically flat manifold to get Euclidean space.

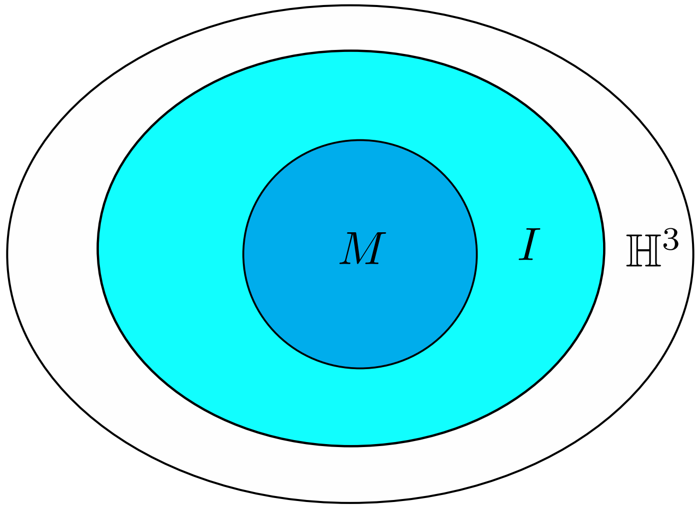

Figure 2.2: Interpolation
I is the interpolation region from B_ρ in M to $\mathbb{H}^3$.

In fact, this computation does not just hold in the time-symmetric, asymptotically flat setting but also in the hyperbolic setting. We will interpolate between an asymptotically hyperbolic manifold and hyperbolic space within an annulus $M_{2r} \backslash M_r$. A formal definition of the interpolated metric $\breve{g}$ and second fundamental form $\check{k}$ is in Section 2.5.2, line (2.458). With a sightly abuse of notation, we still use g, k and ∇ to represent the metric, second fundamental form and connection in the annular region throughout this section, except for Section 2.5.2. Hence it suffices to show that

$$\frac{1}{16\pi} \int_{A_{r,2r}} \mu|\nabla u| + \langle J, \nabla u \rangle dV \to -E - \langle P, x \rangle, \tag{2.47}$$

and the mass formula

$$0 \leq \frac{1}{16\pi} \int_M \frac{|\nabla^2 u + k|\nabla u||^2}{|\nabla u|} + \mu|\nabla u| + \langle J, \nabla u \rangle dV. \tag{2.48}$$

It turns out that it is relatively easy to compute equation (2.47) and that equation (2.48) is equivalent to showing that the integral $\int_{\partial M_r}(\nabla_v|\nabla u| + \nabla_i u k_{iv})dA + \int_M |\nabla u| R_\Sigma dV$ is going to zero in a manifold which is hyperbolic near infinity. Hence, we have to perform the computation for the mass formula only in this very special case.

Remaining difficulties

Even, with the significant simplification from the interpolation method, we still have to deal with the difficulties posed by the nonlinear PDE $\Delta u = -K|\nabla u|$ which we listed in Section 2.3. This presents a huge challenge and requires careful analysis which is performed in Section 2.3 and leads to an expansion for u of the form

$$u = v + |v|^{\frac{3}{2}} r^{-\frac{3}{2}}(A + Br^{-3}) + \text{lower order terms}, \tag{2.49}$$

where $v = -x - t$ is the exact solution the spacetime Laplace equation in hyperbolic space, and A, B are functions on S^2. Note that this formula is in strong contrast to the asymptotically flat setting where the expansion has a much simpler expression and we do not have to analyze the solution up to such lower order terms. The higher order expansion turns out to be absolutely crucial when computing the mass formula (2.48) which even with this expansion turns out to be a highly subtle computation in Section 2.6.

2.3 Existence of the spacetime harmonic equation

This section is to show the existence of the spacetime harmonic equation in the asymptotically hyperbolic setting. We also establish a barrier for the solution.

Theorem 6. *Let (M, g, k) be an three dimensional, complete, simply connected, asymptotically hyperbolic manifold of order τ, $\frac{3}{2} < \tau \leq 3$. Then there exists a solution u of the spacetime harmonic equation*

$$\Delta u + K|\nabla u| = 0 \qquad on \qquad M, \tag{2.50}$$

such that

$$u = v + O_2(1) \quad as \quad r \to \infty. \tag{2.51}$$

Furthermore, we have

$$|u - v| \leq C\frac{v^a}{t^a} \tag{2.52}$$

for some universal constant C, $t = \sqrt{1 + r^2}$ and $a = \min(\frac{3}{2}, \frac{\tau+1}{2})$.

Let $w = u - v$, then w satisfies

$$Lw := \Delta w + \frac{K\nabla(w + 2v)}{|\nabla(w + v)| + |\nabla v|} \cdot \nabla w = f, \tag{2.53}$$

where $f = -\Delta v - K|\nabla v|$. Let $\bar{K} = \frac{K\nabla(w+2v)}{|\nabla(w+v)|+|\nabla v|}$.

Here are the major steps in this section.

1. We use Leray-Schauder fixed point theorem to get the existence of solution w_ρ on a bounded domain $M_\rho := \{x|r(x) \leq \rho\}$, as [HKK20]. Then we need a uniform C^0 estimate of w_ρ so that we can take a subsequence of $\{w_\rho\}$ to converge to the solution

on M.

2. We find a super solution $C(2|v|^a t^{-a} - |v|t^{-\tau})$ to the equation. To show this super solution is a barrier, we need a uniformly C^0 bounded for the solutions w_ρ on a fixed compact set Ω.

3. To get the uniformly C^0 bounds of w_ρ on a fixed compact set, we first construct a subsolution $\underline{\phi}$ only depends on v in subsection 2.3.3. To construct a barrier bounded from below, we solve the equation outside a horosphere $\{v = -\varepsilon\}$, then we glue the solution with $\underline{\phi}$ along the horosphere $\{v = -\varepsilon\}$.

2.3.1 Existence of the solution on a compact set

Proposition 7. *There exists a solution u_ρ that solving the equation with Dirichlet condition*

$$\begin{cases} \Delta u_\rho = -K|\nabla u_\rho| & \text{on } M_\rho, \\ u_\rho = v & \text{on } \partial M_\rho. \end{cases} \tag{2.54}$$

The proof of this theorem is based on [HKK20].

Proof. Let $w_\rho = u_\rho - v$, then w_ρ satisfies

$$\Delta w_\rho + \frac{K\nabla(w_\rho + 2v)}{|w_\rho + v| + |v|} \cdot \nabla w_\rho = f,\ w_\rho|_{\partial M_\rho} = 0. \tag{2.55}$$

We apply Leray-Schauder's fixed point theorem [GT15, Theorem 11.3] to the family of operators

$$\mathcal{F}(w, \sigma) = \sigma \Delta^{-1}\left[-K\left(\frac{\nabla(w + 2v)}{|\nabla(w + v)| + |\nabla v|}\right) \cdot \nabla w - f\right]. \tag{2.56}$$

to obtain a fixed point $\mathcal{F}(w_\rho, 1) = w_\rho$ from a-priori estimates for $\mathcal{F}(\cdot, \sigma)$, $0 \le \sigma \le 1$. Note that $\mathcal{F}(\cdot, 1)$ is a compact operator on $C_0^{1,\alpha}(M_\rho) \to C_0^{1,\alpha}(M_\rho)$. Such obtained fixed point w_ρ solves equation (2.55) which implies that $u_\rho := v + w_\rho$ is spacetime harmonic. More details can be found in [HKK20], Section 4. □

2.3.2 Supersolution

This section is to verify $\Psi = C(2|v|^a t^{-a} - |v|t^{-\tau})$, $a = \min\{\frac{3}{2}, \frac{\tau+1}{2}\}$ is a supersolution of w, where $w = u - v$.

Assume $\bar{u} = v + \Psi$. Since $|g - b| = O_2(t^{-\tau})$, then

$$\Delta\bar{u} - \bar{\Delta}\bar{u} = O(t^{-\tau})(|\bar{\nabla}^2\bar{u}| + |\bar{\nabla}\bar{u}|) = O(|v|t^{-\tau}), \tag{2.57}$$

and

$$|\nabla\bar{u}| - |\bar{\nabla}\bar{u}| = O(t^{-\tau})|\bar{\nabla}\bar{u}| = O(|v|t^{-\tau}). \tag{2.58}$$

As $K - 3 = O(r^{-\tau})$, then

$$\Delta\bar{u} + K|\nabla u| = \bar{\Delta}\bar{u} + 3|\bar{\nabla}\bar{u}| + O(|v|t^{-\tau}). \tag{2.59}$$

Therefore, we need to show there exists C such that

$$\bar{\Delta}(v + \Psi) + 3|\bar{\nabla}(v + \Psi)| - C_0|v|t^{-\tau} > 0, \tag{2.60}$$

where C_0 is a given constant from the error term $O(|v|r^{-\tau})$ in Equation (2.59). Let $\Psi_0 = |v|^{-a}t^a\Psi$, then $\Psi_0 = C(2 - |v|^{1-a}t^{a-\tau})$ and $u = v + |v|^a t^{-a}\Psi_0$. We apply the linear expansion in the appendix, then

$$|v|^{-a}t^{a}\left[\bar{\Delta}\bar{u}+3|\bar{\nabla}\bar{u}|\right] \tag{2.61}$$

$$=\bar{\Delta}\Psi_0-\left[2a\frac{\bar{\nabla}t}{t}-(2a-3)\frac{\bar{\nabla}|v|}{|v|}\right]\cdot\bar{\nabla}\Psi_0 \tag{2.62}$$

$$-\left[(3a-2a^2)|v|^{-1}t^{-1}+a(a+1)t^{-2}\right]\Psi_0 \tag{2.63}$$

$$+O(|v|^{a-2}t^{-a-1}+|v|^{1-a}t^{a-2\tau}+|v|^{1-a}t^{a-\tau-2}+|v|^{a-1}t^{-\tau-1}), \tag{2.64}$$

where we use the last error term to denote the higher order terms in the linear expansion.

Since

$$\bar{\Delta}(|v|^{1-a}t^{a-\tau}) \tag{2.65}$$

$$=\bar{\nabla}\cdot[(1-a)|v|^{-a}t^{a-\tau}\bar{\nabla}|v|+(a-\tau)|v|^{1-a}t^{a-\tau-1}\bar{\nabla}t] \tag{2.66}$$

$$=(1-a)|v|^{-a}t^{a-\tau}\bar{\Delta}|v|+a(a-1)|v|^{-a-1}t^{a-\tau}|\bar{\nabla}|v||^2 \tag{2.67}$$

$$+2(1-a)(a-\tau)|v|^{-a}t^{a-\tau-1}\bar{\nabla}|v|\cdot\bar{\nabla}t \tag{2.68}$$

$$+(a-\tau)(a-\tau-1)|v|^{1-a}t^{a-\tau-2}|\bar{\nabla}t|^2+(a-\tau)|v|^{1-a}t^{a-\tau-1}\bar{\Delta}t \tag{2.69}$$

$$=[3(1-a)+a(a-1)+2(1-a)(a-\tau)+(a-\tau)(a-\tau-1)+3(a-\tau)] \tag{2.70}$$

$$\cdot|v|^{1-a}t^{a-\tau}+2(a-1)(a-\tau)|v|^{-a}t^{a-\tau-1}-(a-\tau)(a-\tau-1)|v|^{1-a}t^{a-\tau-2} \tag{2.71}$$

$$=(3-4\tau+\tau^2)|v|^{1-a}t^{a-\tau}+2(a-1)(a-\tau)|v|^{-a}t^{a-\tau-1} \tag{2.72}$$

$$-(a-\tau)(a-\tau-1)|v|^{1-a}t^{a-\tau-2}, \tag{2.73}$$

and we have

$$\left[2a\frac{\bar{\nabla}t}{t} - (2a-3)\frac{\bar{\nabla}|v|}{|v|}\right] \cdot \bar{\nabla}(|v|^{1-a}t^{a-\tau}) \tag{2.74}$$

$$=[2a(1-a) + 2a(a-\tau) - (2a-3)(1-a) - (2a-3)(a-\tau)]|v|^{1-a}t^{a-\tau} \tag{2.75}$$

$$+ [2a(1-a) - (2a-3)(a-\tau)]|v|^{-a}t^{a-\tau-1} \tag{2.76}$$

$$=3(1-\tau)|v|^{1-a}t^{a-\tau} + (5a - 4a^2 - 2a\tau + 3\tau)|v|^{-a}t^{a-\tau-1}. \tag{2.77}$$

Therefore,

$$|v|^{-a}t^a \left(\bar{\Delta}\bar{u} + 3|\bar{\nabla}\bar{u}|\right) \tag{2.78}$$

$$=C(\tau-\tau^2)|v|^{1-a}t^{a-\tau} - C(6a^2 - 7a - \tau)|v|^{-a}t^{a-\tau-1} \tag{2.79}$$

$$-2C\left[(3a-2a^2)|v|^{-1}t^{-1} + a(a+1)t^{-2}\right] \tag{2.80}$$

$$+ O(|v|^{a-2}t^{-a-1} + |v|^{1-a}t^{a-2\tau} + |v|^{1-a}t^{a-\tau-2} + |v|^{a-1}t^{-\tau-1}). \tag{2.81}$$

If $a = \frac{3}{2}$, and $2 \leq \tau \leq 3$, then $6a^2 - 7a - \tau = 3 - \tau \geq 0$. Therefore, we have

$$|v|^{-a}t^a \left(\bar{\Delta}\bar{u} + 3|\bar{\nabla}\bar{u}|\right) \tag{2.82}$$

$$\leq C(\tau-\tau^2)|v|^{-\frac{1}{2}}t^{\frac{3}{2}-\tau} + O(|v|^{-\frac{1}{2}}t^{-\frac{5}{2}} + |v|^{-\frac{1}{2}}t^{\frac{3}{2}-2\tau} + |v|^{-\frac{1}{2}}t^{-\frac{1}{2}-\tau} + |v|^{\frac{1}{2}}t^{-\tau-1}) \tag{2.83}$$

$$\leq -C_0|v|^{-\frac{1}{2}}t^{\frac{3}{2}-\tau}, \tag{2.84}$$

where the last inquality is from that t is sufficiently large, and we also pick C sufficiently large.

Let $a = \frac{\tau+1}{2}$, and $\tau < 2$, suppose τ satisfies $6a^2 - 7a - \tau < 0$, otherwise, line (2.79) is negative implies line (2.78) is negative. Then we use the first term in (2.80) to control the second term in (2.79), as $|v| \geq \frac{1}{2}t^{-1}$, we have

$$(6a^2 - 7a - \tau)|v|^{-a}t^{a-\tau-1} + 2(3a - 2a^2)|v|^{-1}t^{-1} \tag{2.85}$$

$$=|v|^{-1}t^{-1}[(6a^2 - 7a - \tau)|v|^{\frac{1-\tau}{2}}t^{\frac{1-\tau}{2}} + 2(3a - 2a^2)] \tag{2.86}$$

$$\geq|v|t^{-1}[\sqrt{2}(6a^2 - 7a - \tau) + 2(3a - 2a^2)]. \tag{2.87}$$

When $a = \frac{\tau+1}{2}$, then the solutions to the equation $\sqrt{2}(6a^2 - 7a - \tau) + 2(3a - 2a^2) = 0$ are $\tau \approx 1.4944, -0.4944$. Since $\tau > \frac{3}{2}$, then Line (2.87) is positive. Therefore, for any given C_0, we can find C such that

$$|v|^{-a}t^{a}\left(\bar{\Delta}\bar{u} + 3|\bar{\nabla}\bar{u}|\right) \tag{2.88}$$

$$\leq C(\tau - \tau^2)|v|^{1-a}t^{a-\tau} + O(|v|^{a-2}t^{-a-1} + |v|^{1-a}t^{a-2\tau} + |v|^{1-a}t^{a-\tau-2} + |v|^{a-1}t^{-\tau-1}) \tag{2.89}$$

$$\leq C_0|v|^{1-a}t^{a-\tau}. \tag{2.90}$$

Hence, by choosing a sufficiently large C, Ψ is a supersolution of $w = u - v$.

Since the linear part in Line (2.62) and (2.63) is the main part of the spactime harmonic equation, then we can follow the previous computation to get $-\Psi$ is a subsolution. Here is the lemma to conclude the construction in this subsection.

Lemma 8. *Let $\Psi = C(2|v|^a t^{-a} - |v|t^{-\tau})$, $a = \min\{\frac{3}{2}, \frac{1+\tau}{2}\}$, then $\Psi \backslash -\Psi$ is a supersolution$\backslash$subsolution for w outside a large ball M_ρ, where we pick ρ sufficiently large and w satisfies the equation $\Delta(w + v) + |\nabla(w + v)| = 0$.*

2.3.3 A subsolution depends on v

We want to construct a subsolution only depends on v which will be used in the following subsection.

In this subsection, we use the upper half space model:

$$b = \frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2},$$

the map from the hyperboloidal model to the upper half space model is:

$$(x, y, z, t) \to (\frac{2y}{x+t}, \frac{2z}{x+t}, \frac{2}{x+t}),$$

then $v = -x - t = -2/x_3$.

$$\bar{\Delta} = (x_3 \partial_{x_3})^2 - 2x_3 \partial_{x_3} + \bar{\Delta}_\Sigma. \tag{2.91}$$

Let $x_3 = e^\eta$, $\eta \in (-\infty, \infty)$, then $b = d\eta^2 + e^{-2\eta}(dx_1^2 + dx_2^2)$, $\bar{\Delta} = \partial_\eta^2 - 2\partial_\eta + \bar{\Delta}_\Sigma$, $v = -2e^{-\eta}$.

Since $|g - b| = O_2(t^{-\tau})$, and $|K - 3| = O_1(t^{-\tau})$, then we can rewrite the spacetime harmonic equation in the hyperbolic metric,

$$\bar{\Delta} u + 3|\bar{\nabla} u| = t^{-\tau} O(|\bar{\nabla} u| + |\bar{\nabla}^2 u|). \tag{2.92}$$

Ansatz $\underline{u} = v + \underline{\phi}(\eta)$. We assume

$$\partial_\eta \underline{\phi} > 0, |\partial_\eta \underline{\phi}| \leq C_{\underline{\phi}} e^{-\eta} \text{ and } |\partial_\eta^2 \underline{\phi}| \leq C_{\underline{\phi}} e^{-\eta}, \tag{2.93}$$

where $C_{\underline{\phi}}$ is a positive constant. We choose $C_{\underline{\phi}} > 1$. Since $|\bar{\nabla} v| = |v|$, $|\bar{\nabla}^2 v| = 3|v|$, and there exists a constant C such that $|\bar{\nabla}^2\underline{\phi}| \leq CC_{\underline{\phi}}e^{-\eta}$, then

$$
\begin{aligned}
&t^{-\tau}(|\bar{\nabla}\underline{u}| + |\bar{\nabla}^2\underline{u}|) && (2.94)\\
\leq &t^{-\tau}(4|v| + |\bar{\nabla}\underline{\phi}| + |\bar{\nabla}^2\underline{\phi}|) && (2.95)\\
\leq &[(C+1)C_{\underline{\phi}} + 8]t^{-\tau}e^{-\eta} && (2.96)\\
\leq &[(C+1)C_{\underline{\phi}} + 8]e^{-\eta} \cdot 8(e^{\eta} + e^{-\eta})^{-\frac{3}{2}}t^{-\tau+\frac{3}{2}}, && (2.97)
\end{aligned}
$$

we use $t \geq \frac{1}{2}(|v| + |v|^{-1})$ for the last inequality.

Since $\partial_\eta\underline{\phi} > 0$ and $|\bar{\nabla}\eta| = 1$, then

$$
\begin{aligned}
&\bar{\Delta}\underline{u} + 3|\bar{\nabla}\underline{u}| && (2.98)\\
=&3v + \partial_\eta^2\underline{\phi} - 2\partial_\eta\underline{\phi} + 3|\partial_\eta(-2e^{-\eta} + \underline{\phi})| \cdot |\bar{\nabla}\eta| && (2.99)\\
=&3v + \partial_\eta^2\underline{\phi} - 2\partial_\eta\underline{\phi} + 3(2e^{-\eta} + \partial_\eta\underline{\phi}) && (2.100)\\
=&\partial_\eta^2\underline{\phi} + \partial_\eta\underline{\phi}. && (2.101)
\end{aligned}
$$

Therefore, $\underline{\phi}$ is a subsolution for $w = u - v$ outside a large ball, if $\underline{\phi}$ satisfies

$$\partial_\eta^2\underline{\phi} + \partial_\eta\underline{\phi} = \frac{1}{10}C_{\underline{\phi}}e^{-\eta}(e^{\eta} + e^{-\eta})^{-\frac{3}{2}}, \tag{2.102}$$

where the right side of the equation is due to the estimate in (2.97).

Let $\bar{f}(\eta) = (e^{\eta} + e^{-\eta})^{-\frac{3}{2}}$. We assume $\partial_\eta\underline{\phi} \to 0$, when $\eta \to -\infty$, then we have

$$\partial_\eta\underline{\phi} = e^{-\eta}\int_{-\infty}^{\eta} \frac{1}{10}C_{\underline{\phi}}\bar{f}(s)ds. \tag{2.103}$$

We need to estimate $\partial_\eta\underline{\phi}$ to verify the assumptions in Line (2.93), and we also need to show that $\partial_\eta\underline{\phi}$ is integrable.

Since $\bar{f} \le \min\{e^{\frac{3}{2}\eta}, e^{-\frac{3}{2}\eta}\}$, therefore, when $\eta \le 0$,

$$\partial_\eta \underline{\phi} \le e^{-\eta} \int_{-\infty}^{\eta} \frac{1}{10} C_{\underline{\phi}} e^{\frac{3}{2}s} ds = \frac{1}{15} C_{\underline{\phi}} e^{\frac{1}{2}\eta}. \tag{2.104}$$

When $\eta \ge 0$,

$$\partial_\eta \underline{\phi} \le \frac{1}{10} e^{-\eta} \cdot \left(\int_{-\infty}^{0} C_{\underline{\phi}} e^{\frac{3}{2}s} ds + \int_{0}^{\eta} C_{\underline{\phi}} e^{-\frac{3}{2}s} ds \right) \tag{2.105}$$

$$= \frac{C_{\underline{\phi}}}{10} e^{-\eta} \cdot \left(\frac{2}{3} + \frac{2}{3} - \frac{2}{3} e^{-\frac{3}{2}\eta} \right) \tag{2.106}$$

$$\le \frac{1}{2} C_{\underline{\phi}} e^{-\eta}. \tag{2.107}$$

Therefore, $|\partial_\eta \underline{\phi}| \le \frac{1}{2} C_{\underline{\phi}} e^{-\eta}$. Then we can use the ODE for $\underline{\phi}$ to get

$$|\partial_\eta^2 \underline{\phi}| \le |\partial_\eta \bar{\phi}| + \frac{1}{10} C_{\underline{\phi}} e^{-\eta} (e^{\eta} + e^{-\eta})^{-\frac{3}{2}} \le C_{\underline{\phi}} e^{-\eta}. \tag{2.108}$$

Hence, the assumptions in Line (2.93) hold, and $\partial_\eta \underline{\phi}$ is integrable. Therefore, we have

$$\underline{\phi}(\eta) = -\bar{C}_{\underline{\phi}} + \int_{-\infty}^{\eta} \partial_s \underline{\phi}(s) ds = -\bar{C}_{\underline{\phi}} + \int_{-\infty}^{\eta} e^{-s_0} \int_{-\infty}^{s_0} \frac{1}{10} C_{\underline{\phi}} (e^s + e^{-s})^{\frac{3}{2}} ds ds_0$$

is a well defined subsolution of the equation $\Delta w + \bar{K} \cdot \nabla w = f$, cf. (2.53).

2.3.4 Local C^0 estimate

We first construct a solution on $\Omega_1 := \{v \le -\varepsilon\}$, where ε is chosen to be very small, then we glue this solution to the subsolution $\underline{\phi}$ along $\{v = -\varepsilon\}$. Let $\Sigma := \{v = -\varepsilon\}$.

Lemma 9. *On $\Omega_1 = \{v \le -\varepsilon\}$, there exists a solution u_0 for the spacetime harmonic* *equation with the given Dirichlet condition*

$$\Delta u_0 + K|\nabla u_0| = 0,\ u_0|_\Sigma = v. \tag{2.109}$$

Moreover, $|u_0 - v|$ *is bounded on* Ω_1*, and* $|\partial_{\mathbf{n}}(u_0 - v)|$ *is bounded on* Σ*, where* $\mathbf{n}$ *is the inward normal unit vector of* Ω_1 *on* Σ*.*

Proof. Let $\{\Omega_s\}_s$ be a sequence of compact sets that exhausts Ω_1, and $\Omega_s \cap \Sigma \neq \varnothing$, as shown in Figure 2.3. We assume u_s is the solution on Ω_s,

$$\Delta u_s + K|\nabla u_s| = 0,\ u_s|_{\partial\Omega_s} = v. \tag{2.110}$$

Denote $w_s = u_s - v$, then $w_s|_{\partial\Omega_s} = 0$. Since $\Psi = C(2|v|^a t^{-a} - |v|t^{-\tau})$ is a supersolution outside a compact set. We denote this compact set Ω_0, and by choosing ε sufficiently small, we have $\Omega_0 \subset \Omega_1$. Therefore, on $\Omega_s \backslash \Omega_0$, $L(w_s + \Psi) \leq 0$.

Therefore, the minimum point of $w_s + \Psi$ attains on $\partial\Omega_s \cup \Omega_0$. Suppose we do not have a uniformly lower bound for $\{w_s(x)|x \in \Omega_0\}$, then the minimum point of $w_s + \Psi$ lies in Ω_0. Let $x_s \in \Omega_0$ be the minimum point of $w_s + \Psi$ on Ω_s, then $w_s(x_s) \to -\infty$.

Let $\bar{w}_s = \frac{w_s + \Psi}{|w_s(x_s) + \Psi(x_s)|}$, by passing a subsequence, $\bar{w}_s \to \bar{w}_\infty$, $x_s \to x_0$. Then $\bar{w}_\infty$ satisfies $\Delta \bar{w}_\infty + \bar{K}_\infty \cdot \nabla \bar{w}_\infty = 0$, and $\bar{w}_\infty(x_0) = -1$ is the minimum of $\bar{w}_\infty$. Therefore, $\bar{w}_\infty \equiv -1$. However, $\bar{w}_\infty|_\Sigma = 0$. Contradiction!

Then $\{w_s(x)|x \in \Omega_0\}$ has a uniformly lower bound. Therefore, we can solve the equation on Ω_1.

From the barrier Ψ, we have $|u_0 - v|$ is bounded. Then we apply the standard boundary estimate in elliptic PDE, therefore, $|\partial_{\mathbf{n}}(u_0 - v)|$ is bounded. $\square$

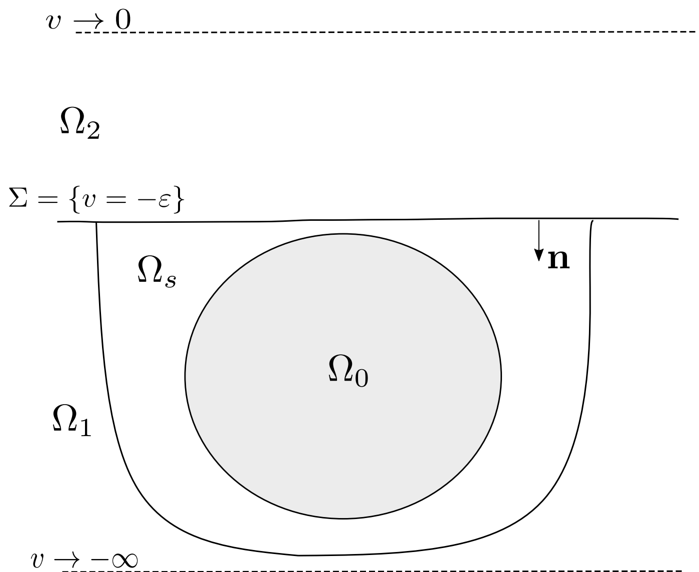

Figure 2.3: Barrier Region

Let $\Omega_2 = \{-\varepsilon < v < 0\}$. Since $v = -2e^{-\eta}$, then $\mathbf{n} = -\frac{\nabla\eta}{|\nabla\eta|}$, we have

$$\partial_{\mathbf{n}}\underline{\phi} = -|\nabla\eta|\partial_\eta\underline{\phi} \leq -C\partial_\eta\underline{\phi}, \tag{2.111}$$

where we apply $\partial_\eta\underline{\phi} > 0$ and there exists C such that $|\nabla\eta| \geq C > 0$ on M.

Recall that

$$\underline{\phi} = -\bar{C}_{\underline{\phi}} + C_{\underline{\phi}}\int_{-\infty}^{\eta} e^{-s_0}\int_{-\infty}^{s_0}\frac{1}{10}(e^s + e^{-s})^{-\frac{3}{2}}ds ds_0.$$

As $|\partial_{\mathbf{n}}(u_0 - v)|$ is bounded on Σ, we can choose $C_{\underline{\phi}}$ large enough such that

$$\partial_{\mathbf{n}}\underline{\phi} \leq -C\partial_\eta\underline{\phi} < \partial_{\mathbf{n}}(u_0 - v) \text{ on } \Sigma.$$

Then we can define the barrier function $\underline{\Phi}$ for the lower bound,

$$\underline{\Phi} := \begin{cases} \underline{\phi}, \text{ on } \Omega_2; \\ \underline{\phi}_0 := u_0 - v + \underline{\phi}(-\ln(\frac{\varepsilon}{2})), \text{ on } M\backslash\Omega_2, \end{cases}$$

where $-\ln(\frac{\varepsilon}{2})$ is the value of $\eta = -\ln\frac{|v|}{2}$ on $\Sigma = \{v = -\varepsilon\}$. We add $\underline{\phi}(-\ln(\frac{\varepsilon}{2}))$ to $u_0 - v$ so that $\underline{\Phi}$ is continuous.

From the choice of $C_{\underline{\phi}}$, on Σ, we have $\partial_{\mathbf{n}}\underline{\phi} < \partial_{\mathbf{n}}\underline{\phi}_0$.

Recall that w_r solves the equation $Lw_r = f$ with the boundary condition $w_r|_{\partial M_r} = 0$. Let x_r be the minimum point of $w_r - \underline{\Phi}$. As $L(w_r - \underline{\Phi}) \leq 0$, then $x_r \in \partial M_r \cup \Sigma$.

If $x_r \in \Sigma$, then at x_r, $\partial_{\mathbf{n}}(w_r - \underline{\phi}) \leq 0$ and $\partial_{\mathbf{n}}(w_r - \underline{\phi_0}) \geq 0$, which runs against to $\partial_{\mathbf{n}}\underline{\phi} < \partial_{\mathbf{n}}\phi_0$. Therefore, $x_r \in \partial M_r$, then for any $p \in M_r$,

$$w_r(p) \geq \underline{\Phi}(p) + w_r(x_r) - \underline{\Phi}_r(x_r) = \underline{\Phi}(p) - \underline{\Phi}(x_r) \geq -2\sup|\underline{\Phi}|. \tag{2.112}$$

Therefore, w_r is uniformly bounded from below.

Since $u_r = w_r + v$ satisfies $\Delta u_r + K|\nabla u_r| = 0$, by the maximum principle, $u_r \leq \max\{u_r(p)|p \in \partial M_r\} \leq 0$. Therefore, $w_r \leq |v|$. For any fixed compact set Ω, we have w_r is uniformly bounded from above.

We conclude this subsection with the lemma below.

Lemma 10. *For any fixed compact set Ω, w_r is uniformly bounded, where w_r is the solution of the equation below*

$$\Delta(w_r + v) + K|\nabla(w_r + v)| = 0, \text{ and } w_r|_{\partial M_r} = 0. \tag{2.113}$$

2.3.5 Barrier

This section is to show that the super solution Ψ can be used to construct a barrier.

We first solve the equation on a fixed compact set $\Omega \subset M$, then patching it with the super solution ψ. Let $\tilde{w}$ be the solution satisfying the equation with Dirichlet condition

$$\Delta \tilde{w} + \nabla \tilde{w} \cdot \frac{K\nabla(\tilde{w} + 2v)}{|\nabla(w+v)| + |\nabla v|} = f \text{ on } M_{\rho_0};\ \tilde{w} = \Psi + C \text{ on } S_{\rho_0}. \tag{2.114}$$

Lemma 11. *Suppose there exists a fixed compact set $\Omega \subset M$ such that w_ρ is uniformally bounded by C_Ω, then there exists C such that*

$$\bar{w} = \begin{cases} \tilde{w} & on\ M_{\rho_0} \setminus \Omega, \\ \Psi + C & on\ M \setminus M_{\rho_0} \end{cases} \tag{2.115}$$

serves a barrier for all w_ρ. Therefore, w_ρ is uniformly bounded.

Proof. We need to show that there exists C such that $\nabla_\nu \tilde{w} - \nabla_\nu \Psi > 0$ on S_{r_0}, where ν is the outer normal vector on ∂M_{r_0}. Suppose there does not exist such a C, then we have a sequence $\{\tilde{\omega}_i\}$ such that $\tilde{w}_i$ solves the equation with boundary value $\Psi + C_i$ on S_{r_0}, and there exists $x_i \in S_{r_0}$ such that $\nabla_\nu \tilde{w}_i(x_i) \le \nabla_\nu \Psi(x_i)$. Let $w_i = C^{-1}\tilde{w}_i$, then $w_i \to 1$ on S_{r_0} and $w_i \to 0$ on $\partial\Omega$. According to Schauder estimate, we have $\|w_i\|_{C^{2,\alpha}(M_{r_0}\setminus\Omega)} \le \tilde{C}$, where $\tilde{C}$ only depends on the domain $M_{r_0}\setminus\Omega$ and f.

Then w_i subsequently converges to w_∞, and w_∞ is a solution of the equation with boundary value $w_\infty = 1$ on S_{r_0} and $w_\infty = 0$ on $\partial\Omega$. By passing a subsequence, we have $x_i \to x_0$ on S_{r_0}, then $\nabla_\nu w_\infty(x_0) \le 0$, since $\nabla_\nu \Psi(x_i)$ is bounded on S_{r_0}. However, this violates Hopf's Lemma as $w_\infty(x_0)$ is the maximum on $M_{r_0}\setminus\Omega$.

Hence, we have proven $\nabla_\nu \tilde{w} - \nabla_\nu \Psi > 0$ on S_{r_0}. Then we can follow the same augment in [HKK20, Section 4.2] or Lemma 9 to show that $\bar{w}$ is a barrier. $\square$

Since Ψ is positive and w_ρ is uniform bounded on compact set, then there exists $\hat{C}$ such that

$$\hat{C}\Psi \geq w_\rho \text{ on } M_{r_0}. \tag{2.116}$$

By definition $w_\rho = 0$ on S_ρ, then $\hat{C}\Psi \geq w_\rho$ on S_r. Since Ψ is a supersolution, then we have $|w_\rho| \leq \hat{C}\Psi$ on M_r.

Therefore, we can solve the spacetime harmonic equation on M by taking the limit of a subsequence of $\{w_\rho\}$, and we also have $|w| \leq \hat{C}\Psi$. Then we have proven the main Theorem 6 in this section.

2.4 The expansion of u

In this section, we compute the expansion of our spacetime harmonic function u. Since $g-b$, $k-g$ have different decay rates in the purely hyperbolic and the interpolation region, we have different expansions in these two regions. More precisely, we have $g=k=b$ and $|g-b|_b = O_{2,\alpha}(r^{-\tau})$, $|k-b|_b = O_{1,\alpha}(r^{-\tau})$ for some $\frac{3}{2} < \tau \leq 3$, according to the definition of the general asymptotics. Note that even though the interpolation region is contained in a compact set (where one would usually not speak of decay rates), we still have to treat it as non-compact, since our interpolation annulus is moving to infinity and our estimates have to be uniform for all annuli.

Theorem 12. *Let u be a solution to the spacetime Laplace equation $\Delta u = -K|\nabla u|$ with $u \to v$ at ∞. Then $w = u - v$ satisfies:*

1. *In the purely hyperbolic region, we have $w = O_{i,\alpha}(|v|^{\frac{3}{2}} r^{-\frac{3}{2}})$, for any $i \in \mathbb{N}$. Moreover, $w = |v|^{\frac{3}{2}} r^{-\frac{3}{2}} \psi$, where*

$$\psi = A + \frac{B}{r^2} + \hat{\psi} \tag{2.117}$$

with A, B being functions on S^2, $A \in H^{3-\varepsilon}(S^2)$, $6B = \frac{15}{4}A - \Delta_{S^2} A$, and $\hat{\psi}$ being a lower order term satisfying

$$\|\hat{\psi}\|_{L^2(S^2)} = O(r^{-3+\varepsilon}),\ \|\partial_r \hat{\psi}\|_{H^{-1}(S^2)} = O(r^{-4+\varepsilon}),\ \|\partial_r^2 \hat{\psi}\|_{H^{-2}(S^2)} = O(r^{-5+\varepsilon}), \tag{2.118}$$

for any $\varepsilon > 0$. Furthermore, away from $\theta = \pi$ (where $v(r,\theta,\phi) \to 0$ for $r \to \infty$), A is smooth and $\hat{\psi} = O_{2,\alpha}(r^{-3})$.

2. *In the interpolation region, we have $w = O_{3,\alpha}(|v|^a r^{-a})$ for $a = \min\{\frac{\tau+1}{2}, \frac{3}{2}\}$.*

Moreover, we have $w = |v|r^{-1}\phi$*, where*

$$\phi = \hat{A} + \hat{\phi}, \tag{2.119}$$

with $\hat{A} \in H^{\tau-\varepsilon-1}$ *and* $\hat{\phi}$ *satisfying*

$$\int_{S^2} \hat{\phi}^2 d\sigma = O(r^{-2\tau}), \ \int_{S^2} (\partial_r \hat{\phi})^2 d\sigma = O(r^{-2\tau}). \tag{2.120}$$

In order to prove the first part of this result, we begin with using the C^0 barrier estimates for w which we established in the previous section, to obtain higher order estimates for w. Then we can linearly expand the equation for ψ

$$r^2 \partial_r^2 \psi + r^{-2} \Delta_{S^2} \psi = h, \tag{2.121}$$

where h is an error term which will be precisely defined in Line 2.144. The key idea regarding this expansion is that it allows us to use separation of variables. More precisely, the linear expansion allows us to split the PDE into a linear part and a non-linear error term h. Thus, we can use separation of variables even in this non-linear setting as long as we are able to successfully estimate the error term h. This separation of variables argument then allows us to reduce the original PDE (2.121) on M into ODEs.

For the separation of variables argument, we make the Ansatz

$$\psi = \sum_{i=0}^{\infty} a_i(r) \chi_i, \tag{2.122}$$

where a_i are a family of radial functions, and χ_i are spherical harmonics on S^2 (with respect to the round metric). Let us denote with A_i the leading order (i.e. constant in r) term of $a_i(r)$ which then allows us to define A, the leading order term of the expansion for ψ

$$A = \sum_{i=0}^{\infty} \mathcal{A}_i \chi_i. \tag{2.123}$$

Next, we establish the regularity for the highest order term A and estimate $\tilde{\psi} := \psi - A$ via a bootstrap argument. This allows us to extract the highest order term Br^{-2} from $\tilde{\psi}$, i.e. the second highest order term from ψ, and finish the proof of expansion in the purely hyperbolic region.

We apply a similar argument to establish the second part of Theorem 12, i.e. the expansion in the interpolation region. The difference is that instead of the barrier $C|v|^{\frac{3}{2}}t^{-\frac{3}{2}}$, we only have the barrier $C(2|v|^a t^{-a} - |v|t^{-\tau})$ in this region. Another difference is that the metric g is not purely hyperbolic, only $C^{2,\alpha}$ close to the hyperbolic metric which leads to less regularity.

2.4.1 Higher order regularity of w

In this section, we prove that w satisfies good decay estimates: up to arbitrary orders in the purely hyperbolic region and up to the third order in the annular interpolation region. We begin with a $W^{2,p}$ estimate of w.

Proposition 13. *Let $w = u - v$ be the solution of*

$$\Delta w + K \frac{\nabla(w + 2v)}{|\nabla(w + v)| + |\nabla v|} \cdot \nabla w = f, \tag{2.124}$$

where $f = -\Delta v - K|\nabla v|$. Then we have, for any $1 < p < \infty$, $1 < a \leq \frac{3}{2}$

$$\|w\|_{W^{2,p}_{loc}} = O(|v|^a t^{-a}) \tag{2.125}$$

in the purely hyperbolic region, and

$$\|w\|_{W^{2,p}_{loc}} = O(|v|^a t^{-a} + |v| t^{-\tau}) \tag{2.126}$$

in the annular interpolation region. In particular, for $1 \le a \le \frac{1}{2} + \frac{\tau}{2}$*, we obtain in the latter case*

$$\|w\|_{W^{2,p}_{loc}} = O(|v|^a t^{-a}). \tag{2.127}$$

Recall that $f = 0$ in the purely hyperbolic region and that $f = O_{1,\alpha}(|v| r^{-\tau})$ in the annular interpolation region.

Proof. We prove the statement in the purely hyperbolic region. The estimate in the annular interpolation region is obtained analogously. Recall that we showed in the previous barrier section that $w = O(|v|^a t^{-a})$ for any $a \le \frac{3}{2}$. To prove the above proposition, it suffices to show $\||v|^a t^{-a}\|_{L^p_{loc}} = O(|v|^a t^{-a})$ since this allows us to apply elliptic estimates. For this purpose, we fix a point q_0 outside a large ball $B(0, r)$. Let $q \in B(q_0, 1)$ and denote with $\gamma(s) : [0, d(q_0, q)] \to B(q_0, 1)$ the length minimizing geodesic connecting q_0 and q parameterized by the arc length. Since $|\gamma'(s)| = 1$, we have

$$\left| \frac{d}{ds} v(\gamma(s)) \right| = |\gamma'(s) \cdot (\nabla v)_{\gamma(s)}| \le |\nabla v|_{\gamma(s)} \le |v(\gamma(s))|. \tag{2.128}$$

Therefore, $|v(\gamma(s))| \le e^s |v(\gamma(0))|$, and since $d(q_0, q) \le 1$, we have $|v(q)| \le e|v(q_0)|$. Combining this estimate with our assumption $w = O(|v|^a t^{-a})$, we have the L^p_{loc} estimate $\||v|^a t^{-a}\|_{L^p_{loc}} = O(|v|^a t^{-a})$. Thus, we can use the $W^{2,p}$ estimate for elliptic PDE to obtain (2.125). □

By Morrey's embedding theorem we also get $C^{1,\alpha}$ estimates for for w, i.e. $\|w\|_{C^{1,\alpha}_{loc}} =$

$O(|v|^a t^{-a})$. Next, we would like to use Schauder estimates to obtain $C^{2,\alpha}$ regularity for w. For this purpose, we need a $C^{0,\alpha}$ estimate for $\bar{K} = K\frac{\nabla(w+2v)}{|\nabla(w+v)|+|\nabla v|}$:

Lemma 14. *For $0 < \alpha < 1$, $|\bar{K}|_{C^{0,\alpha}}$ is bounded in both the purely hyperbolic region and the annular interpolation region.*

Proof. According to the $C^{1,\alpha}$ estimate of w, we have $|\nabla(w+2v)|_{C^{0,\alpha}} \leq C_1|v|$ for some fixed constant C_1. Moreover,

$$\Big||\nabla(w+v)| + |\nabla v|\Big|_{C^{0,\alpha}} \leq 2|\nabla v|_{C^{0,\alpha}} + |\nabla w|_{C^{0,\alpha}} \leq C_2|v| \tag{2.129}$$

for a constant C_2. For $p, q \in M$ let us denote with $d(p,q)$ the distance between p and q. We compute

$$\left|\left(\frac{\nabla(w+2v)}{|\nabla(w+v)|+|\nabla v|}\right)_p - \left(\frac{\nabla(w+2v)}{|\nabla(w+v)|+|\nabla v|}\right)_q\right| \tag{2.130}$$

$$=\left|\frac{[\nabla(w+2v)]_p(|\nabla(w+v)|+|\nabla v|)_q - [\nabla(w+2v)]_q(|\nabla(w+v)|+|\nabla v|)_p}{(|\nabla(w+v)|+|\nabla v|)_p\cdot(|\nabla(w+v)|+|\nabla v|)_q}\right| \tag{2.131}$$

$$\leq |[\nabla(w+2v)]_p|\cdot\frac{|(|\nabla(w+v)|+|\nabla v|)_q - (|\nabla(w+v)|+|\nabla v|)_p|}{(|\nabla(w+v)|+|\nabla v|)_p\cdot(|\nabla(w+v)|+|\nabla v|)_q} \tag{2.132}$$

$$+\frac{|[\nabla(w+2v)]_p - [\nabla(w+2v)]_q|}{(|\nabla(w+v)|+|\nabla v|)_q} \tag{2.133}$$

$$\leq C_1|v|_p\cdot\frac{C_2|v|_q d(p,q)^\alpha}{|v|_p\cdot|v|_q} + C_1 d(p,q)^\alpha. \tag{2.134}$$

This shows that $\bar{K}$ is bounded in $C^{0,\alpha}$. □

Since $\bar{K}$ is bounded in $C^{0,\alpha}$, we have $|w|_{C^{2,\alpha}} = O(|v|^a r^{-a})$ by Schauder's estimate. But having a better regularity for w implies that we also have a better regularity for $\bar{K}$.

Proposition 15. *1. In the purely hyperbolic region, suppose that $|w|_{C^{k,\alpha}} = O(|v|^a r^{-a})$, where $a \leq \frac{3}{2}$ and $k \in \mathbb{N}$. Then $|\bar{K}|_{C^{k-1,\alpha}}$ is bounded.*

2. In the annular region, suppose that $|w|_{C^{2,\alpha}} = O(|v|r^{-1})$*, then* $|\bar{K}|_{C^{1,\alpha}}$ *is bounded.*

Proof. (1) If $|w|_{C^{k,\alpha}} = O(|v|^a r^{-a})$, then there exists a constant $C > 0$ such that

$$C^{-1}|v| \leq |\nabla(w+v)|_{C^{i,\alpha}} \leq C|v| \tag{2.135}$$

for all $0 \leq i \leq k-1$. Since $|\nabla^j v| = O(|v|)$, we have $|\nabla^j(|\nabla v|)|_{C^{0,\alpha}} = O(|v|)$. Since $|w|_{C^{k,\alpha}} = O(|v|^a r^{-a})$, we have according to the linear expansion $|\nabla(w+v)| = |v| + O_{k-1,\alpha}(|v|r^{-1})$ and

$$C^{-1}|v| \leq |\nabla^i(|\nabla(w+v)|)|_{C^{0,\alpha}} \leq C|v|, \text{ for any } 0 \leq i \leq k-1. \tag{2.136}$$

Thus, the term

$$\left|\frac{\nabla(w+2v)}{|\nabla(w+v)|+|\nabla v|}\right|_{C^{k-1,\alpha}} \leq C|\nabla(w+2v)|_{C^{k-1,\alpha}} \cdot \left|\frac{1}{|\nabla(w+v)|+|\nabla v|}\right|_{C^{k-1,\alpha}} \leq C_1|v| \cdot |v|^{-1} \tag{2.137}$$

is bounded. Hence, $|\bar{K}|_{C^{k-1,\alpha}}$ is bounded, then we obtain $|w|_{C^{k+1,\alpha}} = O(|v|^a r^{-a})$.

(2) In the annular region we have $v = O_{3,\alpha}(|v|)$ and $|\nabla v| = O_{2,\alpha}(|v|)$. Suppose that $|w|_{C^{2,\alpha}} = O(|v|r^{-1})$. This implies

$$C^{-1}|v| \leq |\nabla(|\nabla(w+v)|)|_{C^{0,\alpha}} \leq C|v| \tag{2.138}$$

and

$$|\nabla(w+2v)|_{C^{1,\alpha}} = O(|v|). \tag{2.139}$$

Since $K = \operatorname{Tr}_g k = 3 + O_{1,\alpha}(r^{-\tau})$, we know that

$$|\bar{K}|_{C^{1,\alpha}} = \left| K \frac{\nabla(w+2v)}{|\nabla(w+v)| + |\nabla v|} \right|_{C^{1,\alpha}} \tag{2.140}$$

$$\leq C |\nabla(w+2v)|_{C^{1,\alpha}} \cdot \left| \frac{1}{|\nabla(w+v)| + |\nabla v|} \right|_{C^{1,\alpha}}, \tag{2.141}$$

is bounded. This finishes the proof. □

Hence, we can iterate the above procedure to obtain:

Corollary 16. *The function $w = u - v$ is smooth in the purely hyperbolic region. Moreover, $|w|_{C^{k,\alpha}} = O(|v|^a r^{-a})$ for any $k \geq 0$, $1 \leq a \leq \frac{3}{2}$. In the annular interpolation region we obtain $w = O_{3,\alpha}(|v| r^{-1})$.*

Note that if we had stronger asymptotics such as

$$|h|_b = |g - b|_b = O_{k+1,\alpha}(r^{-\tau}), \; |k - b|_b = O_{k,\alpha}(r^{-\tau}) \tag{2.142}$$

for some $k > 1$, we would obtain stronger estimates for w. More precisely, we would have $w = O_{k+2,\alpha}(|v|^a r^{-a})$ and $1 \leq a \leq \frac{1}{2} + \frac{\tau}{2}$.

2.4.2 Linear expansion of the PDE

Using the $C^{k,\alpha}$ estimates of w from the previous section, we reformulate the PDE for ψ using the linear expansion of Appendix 2.B. Recall that ψ is defined via $w = |v|^a r^{-a} \psi$ where $a = \frac{3}{2}$.

Lemma 17. *We have $\psi = O_k(1)$ for all $k \geq 0$, and ψ satisfies*

$$r^2 \partial_r^2 \psi + r^{-2} \Delta_{S^2} \psi = h \tag{2.143}$$

where $\tilde{h}$ is defined via

$$h = -\partial_r^2\psi - \frac{2}{r}\partial_r\psi + \frac{15}{4}t^{-2}\psi \tag{2.144}$$

$$-\frac{3}{2}|v|^{\frac{1}{2}}t^{-\frac{3}{2}}\left(|\nabla\psi|^2 - \left|\frac{\nabla|v|}{|v|}\cdot\nabla\psi\right|^2\right) - 3|v|^{-1}\psi\nabla(|v|^{\frac{3}{2}}t^{-\frac{3}{2}})\cdot\nabla\psi \tag{2.145}$$

$$+\frac{9}{2}|v|^{-\frac{1}{2}}t^{-\frac{5}{2}}\psi\frac{\nabla|v|}{|v|}\cdot\nabla\psi + \left(\frac{27}{4}|v|^{-\frac{1}{2}}t^{-\frac{5}{2}} - \frac{27}{8}|v|^{-\frac{3}{2}}t^{-\frac{7}{2}}\right)\psi^2 \tag{2.146}$$

$$+\frac{3}{2}|v|t^{-3}|\nabla\psi|^2\frac{\nabla|v|}{|v|}\cdot\nabla\psi + \frac{3}{2}|v|t^{-3}\left(\frac{\nabla|v|}{|v|}\cdot\nabla\psi\right)^3 + O_2(|v|^{\frac{1}{2}}t^{-\frac{7}{2}}). \tag{2.147}$$

and satisfies $h = O(r^{-1})$.

Proof. According to the linear expansion in Appendix 2.B, we have for $a = \frac{3}{2}$

$$\Delta\psi - 3\frac{\nabla t}{t}\cdot\nabla\psi - \frac{15}{4}t^{-2}\psi \tag{2.148}$$

$$= -\frac{3}{2}|v|^{\frac{1}{2}}t^{-\frac{3}{2}}\left(|\nabla\psi|^2 - \left|\frac{\nabla|v|}{|v|}\cdot\nabla\psi\right|^2\right) - 3|v|^{-1}\psi\nabla(|v|^{\frac{3}{2}}t^{-\frac{3}{2}})\cdot\nabla\psi \tag{2.149}$$

$$+\frac{9}{2}|v|^{-\frac{1}{2}}t^{-\frac{5}{2}}\psi\frac{\nabla|v|}{|v|}\cdot\nabla\psi - \left(\frac{27}{4}|v|^{-\frac{1}{2}}t^{-\frac{5}{2}} - \frac{27}{8}|v|^{-\frac{3}{2}}t^{-\frac{7}{2}}\right)\psi^2 \tag{2.150}$$

$$+\frac{3}{2}|v|t^{-3}|\nabla\psi|^2\frac{\nabla|v|}{|v|}\cdot\nabla\psi + \frac{3}{2}|v|t^{-3}\left(\frac{\nabla|v|}{|v|}\cdot\nabla\psi\right)^3 + O_2(|v|^{\frac{1}{2}}t^{-\frac{7}{2}}). \tag{2.151}$$

Observe that

$$\Delta\psi - 3\frac{\nabla t}{t}\cdot\nabla\psi \tag{2.152}$$

$$=(1+r^2)(\partial_r^2\psi + \frac{3}{r}\partial_r\psi) - \frac{\partial_r\psi}{r} - 3r\partial_r\psi + r^{-2}\Delta_{S^2}\psi \tag{2.153}$$

$$=(1+r^2)\partial_r^2\psi + \frac{2}{r}\partial_r\psi + r^{-2}\Delta_{S^2}\psi. \tag{2.154}$$

Combining all the above equations with the definition of h, we obtain

$$r^2\partial_r^2\psi + r^{-2}\Delta_{S^2}\psi = h. \tag{2.155}$$

Finally, we use the $C^{k,\alpha}$ estimates established for w in Corollary 16 to obtain the $C^{k,\alpha}$ estimate $h = O_{k,\alpha}(r^{-1})$ which finishes the proof. □

Similarly, we obtain in the annular interpolation region the following estimate for $\phi = w|v|^{-1}r$.

Lemma 18. *We have* $\phi = O_3(1)$ *and* ϕ *satisfies*

$$\mathfrak{h} = r^2\partial^2\phi + r^{-2}\Delta_{S^2}\phi \tag{2.156}$$

where

$$\mathfrak{h} = -\partial_r^2\phi - \frac{2}{r}\partial_r\phi - \left[\frac{\bar{\nabla}t}{t} - \frac{\bar{\nabla}|v|}{|v|}\right] \cdot \bar{\nabla}\phi + |v|^{-1}t^{-1}\phi + O_1(r^{1-\tau}). \tag{2.157}$$

Proof. Expressing the PDE $\Delta(w+v) + K|\nabla(w+v)| = 0$ with the Laplacian $\bar{\Delta}$ and the gradient $\bar{\nabla}$ with respect to the hyperbolic metric b, we obtain

$$0 = |v|^{-1}r\big[\bar{\Delta}w + 3(|\bar{\nabla}(w+v)|_b - |\bar{\nabla}v|_b) + O_1(r^{-\tau}|v|)\big]. \tag{2.158}$$

Next, recall that the linear expansion states for $a = 1$,

$$
\begin{aligned}
&|v|^{-1}t(\Delta u + 3|\nabla u|) && (2.159)\\
=&\Delta\phi - \left[2\frac{\nabla t}{t} + \frac{\nabla|v|}{|v|}\right] \cdot \nabla\phi - \left[|v|^{-1}t^{-1} + 2t^{-2}\right]\phi && (2.160)\\
&+ \left(3|v|^{-1}t^{-2} - \frac{3}{2}t^{-3} - \frac{3}{2}|v|^{-2}t^{-3}\right)\phi^2 && (2.161)\\
&+ \frac{3}{2}t^{-1}\left(|\nabla\phi|^2 - \left|\frac{\nabla|v|}{|v|} \cdot \nabla\phi\right|^2\right) + 3|v|^{-1}\phi\nabla(|v|^{\frac{3}{2}}t^{-\frac{3}{2}}) \cdot \nabla\phi && (2.162)\\
&- 3|v|^{-1}t^{-2}\phi\frac{\nabla|v|}{|v|} \cdot \nabla\phi + \frac{3}{2}t^{-2}|\nabla\phi|^2\frac{\nabla|v|}{|v|} \cdot \nabla\phi && (2.163)\\
&+ \frac{3}{2}t^{-2}\left(\frac{\nabla|v|}{|v|} \cdot \nabla\phi\right)^3 + O_2(|v|^{-\frac{1}{2}}t^{-\frac{5}{2}}). && (2.164)
\end{aligned}
$$

Inserting this into equation 2.158 and observing that most terms are lower orders, we obtain

$$
0 = \bar{\Delta}\phi - \left[2\frac{\bar{\nabla}t}{t} + \frac{\bar{\nabla}|v|}{|v|}\right] \cdot \bar{\nabla}\phi - |v|^{-1}t^{-1}\phi + O_1(r^{1-\tau}). \quad (2.165)
$$

A further computation yields

$$
\begin{aligned}
0 =&\bar{\Delta}\phi - 3\frac{\bar{\nabla}t}{t} \cdot \bar{\nabla}\phi + \left[\frac{\bar{\nabla}t}{t} - \frac{\bar{\nabla}|v|}{|v|}\right] \cdot \bar{\nabla}\phi - |v|^{-1}t^{-1}\phi + O_1(r^{1-\tau}) && (2.166)\\
=&(1 + r^2)\partial_r^2\phi + \frac{2}{r}\partial_r\phi + r^{-2}\Delta_{S^2}\phi + \left[\frac{\bar{\nabla}t}{t} - \frac{\bar{\nabla}|v|}{|v|}\right] \cdot \bar{\nabla}\phi - |v|^{-1}t^{-1}\phi + O_1(r^{1-\tau}), && (2.167)
\end{aligned}
$$

which finishes the proof. □

2.4.3 Existence and regularity of the first order expansion in the purely hyperbolic region

The goal of this subsection is to show the following result which will be the main ingredient of Theorem 12.

Theorem 19. *We can decompose ψ as $\psi = \tilde{\psi} + A$ where A is a function on S^2 and $\tilde{\psi}$ is a function on M such that for any $\varepsilon > 0$, the following is true:*

1. $A \in H^{3-\varepsilon}(S^2)$.
2. $\sum_{j=1}^{\infty} \tilde{\psi}_j^2 = O(r^{-4+\varepsilon})$ *and* $\sum_{j=1}^{\infty} \lambda_j^d \tilde{\psi}_j^2 = O(r^{-6+2d+\varepsilon})$ *for* $1 \leq d \leq 3 - \varepsilon$.
3. $\sum_{j=1}^{\infty} (\partial_r \tilde{\psi}_j)^2 = O(r^{-6+\varepsilon})$ *and* $\sum_{j=1}^{\infty} \lambda_j^d (\partial_r \tilde{\psi}_j)^2 = O(r^{-8+2d+\varepsilon})$ *for* $1 \leq d \leq 3 - \varepsilon$.
4. $\sum_{j=1}^{\infty} (\partial_r^2 \tilde{\psi}_j)^2 = O(r^{-8+\varepsilon})$ *and* $\sum_{j=1}^{\infty} \lambda_j^d (\partial_r^2 \tilde{\psi}_j)^2 = O(r^{-10+2d+2\varepsilon})$ *for* $1 \leq d \leq 2 - \varepsilon$.

As a corollary, we also obtained improved C^2 control on ψ and thus w, cf. Corollary 31. This will be important in the boundary integral computation in Section 2.6.2.

The proof of Theorem 19 is quite complicated so we give here a brief overview of the argument. We begin in Section 2.4.3 with a separation of variables argument which allows us to split up ψ into the higher order term A (which is a function on S^2) and the lower order term $\tilde{\psi}$. To establish regularity for $\tilde{\psi}$ and A, we use a bootstrap argument. We show that regularity for $\tilde{h}$ implies regularity for ψ, see Section 2.4.3, and that regularity for ψ implies regularity for $\tilde{h}$, see Section 2.4.3. In 2.4.3 we complete the bootstrap argument and finish the proof of Theorem 19.

Separation of variables and the construction of the expansion

We would like to apply separation of variables to analyze the asymptotics of our function ψ. This will allow us to decompose ψ as $\psi = A + \tilde{\psi}$ where the leading order term A is a function on S^2. For the precise definition of A and $\tilde{\psi}$ see the end of this subsection.

Let $\{\chi_i\}_{i=0}^{\infty}$ denote an orthonormal basis of spherical harmonics on S^2. This means that $\Delta_{S^2}\chi_i = -\lambda_i\chi_i$ on S^2 with eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \cdots \to +\infty$. Spherical harmonics are obtained from restricting homogeneous harmonic polynomials on $\mathbb{R}^3$ to S^2. In particular, $\chi_0 = c_0$, $\chi_1 = c_1 x|_{S^2}$, $\chi_2 = c_1 y|_{S^2}$, $\chi_3 = c_1 z|_{S^2}$, where c_0, c_1 are normalizing constants. Recall that we can use the spherical harmonics to define Sobolev spaces $H^s(S^2)$ - even for non-integer s - which we will use in the following repeatedly without further reference.

Proposition 20. *We have $u \in H^s(S^2)$, $s \in \mathbb{R}$, if and only if*

$$\sum_{i\neq 0} \lambda_i^s \langle u, \chi_i\rangle^2_{L^2(S^2)} < \infty \text{ and } u \in L^2(S^2), \text{ when } s \geq 0; \tag{2.168}$$

$$\sum_{i\neq 0} \lambda_i^s \langle u, \chi_i\rangle^2_{L^2(S^2)} < \infty, \text{ when } s < 0. \tag{2.169}$$

Using (θ, φ) coordinates on S^2, we write

$$\psi(r, \theta, \varphi) = \sum_{i=0}^{\infty} a_i(r)\chi_i(\theta, \varphi). \tag{2.170}$$

Plugging this into equation (2.121) yields

$$r^2 \sum_{i=0}^{\infty} a_i''\chi_i = h + r^{-2} \sum_{i=0}^{\infty} \lambda_i a_i \chi_i. \tag{2.171}$$

We would like to consider this as a family of ODEs for the radial functions a_i on $M \setminus M_{r_0}$, $r_0 \gg 1$. We multiply by $\frac{\chi_j}{r^2}$ and integrate over S^2 to find

$$a_j'' - \frac{\lambda_j}{r^4} a_j - r^{-2} h_j(r) = 0. \tag{2.172}$$

Since ψ, $|\nabla\psi|$ are bounded, we have $a_0(r) = O(1)$, $a_0'(r) = O(r^{-1})$. According to the barrier $v^a t^{-a}$ and using the fact $\lambda_0 = 0$, the solution for Equation (2.172) is

$$a_0 = D_0 + \int_r^\infty \left[\int_s^\infty \frac{h_0(t)}{t^2} dt \right] ds, \tag{2.173}$$

where D_0 is a bounded function on S^2 depending on the initial conditions. Next, we would like to solve the ODE for a_j, $j \geq 1$. For this purpose we first observe that the fundamental solutions of the homogeneous equation

$$a'' - \frac{\lambda_j}{r^4} a = 0 \tag{2.174}$$

are

$$a = L_j r \cosh\left(\frac{\sqrt{\lambda_j}}{r} \right) + D_j r \sinh\left(\frac{\sqrt{\lambda_j}}{r} \right) \tag{2.175}$$

for some functions L_j, D_j are functions on S^2. To solve the inhomogeneous equation, we compute the Wronskian w_j:

$$w_j = \det \begin{pmatrix} r\cosh\left(\frac{\sqrt{\lambda_j}}{r}\right) & r\sinh\left(\frac{\sqrt{\lambda_j}}{r}\right) \\ \cosh\left(\frac{\sqrt{\lambda_j}}{r}\right) - \frac{\sqrt{\lambda_j}}{r}\sinh\left(\frac{\sqrt{\lambda_j}}{r}\right) & \sinh\left(\frac{\sqrt{\lambda_j}}{r}\right) - \frac{\sqrt{\lambda_j}}{r}\cosh\left(\frac{\sqrt{\lambda_j}}{r}\right) \end{pmatrix} = -\sqrt{\lambda_j}. \tag{2.176}$$

Thus, the solutions to the non-homogeneous equation (2.172) is

$$a_j = L_j r \cosh\left(\frac{\sqrt{\lambda_j}}{r}\right) + D_j r \sinh\left(\frac{\sqrt{\lambda_j}}{r}\right) \tag{2.177}$$

$$+ r \cosh\left(\frac{\sqrt{\lambda_j}}{r}\right) \int_r^\infty s \sinh\left(\frac{\sqrt{\lambda_j}}{s}\right) \frac{h_j(s)}{s^2 w_j(s)} ds \tag{2.178}$$

$$- r \sinh\left(\frac{\sqrt{\lambda_j}}{r}\right) \int_r^\infty s \cosh\left(\frac{\sqrt{\lambda_j}}{s}\right) \frac{h_j(s)}{s^2 w_j(s)} ds. \tag{2.179}$$

Note that the radial functions in front of the coefficients L_j are blowing up for $r \to \infty$. Since $|a_j| \leq C$, we thus have $L_j = 0$. Simplifying (2.178) and (2.179), we obtain

$$a_j = D_j r \sinh\left(\frac{\sqrt{\lambda_j}}{r}\right) + \frac{r}{\sqrt{\lambda_j}} \int_r^\infty \sinh\left(\frac{\sqrt{\lambda_j}}{r} - \frac{\sqrt{\lambda_j}}{s}\right) \cdot \frac{h_j(s)}{s} ds. \tag{2.180}$$

According to the boundary condition

$$\langle \psi(r_0), \chi_j \rangle = a_j(r_0), \quad \lim_{r\to\infty} a_j(r) = 0, \tag{2.181}$$

we have

$$D_j = \frac{1}{r_0 \sinh(\sqrt{\lambda_j}/r_0)} \left(1 - \frac{r_0}{\sqrt{\lambda_j}} \int_{r_0}^\infty \sinh\left(\frac{\sqrt{\lambda_j}}{r_0} - \frac{\sqrt{\lambda_j}}{s}\right) \cdot \frac{h_j(s)}{s} ds\right). \tag{2.182}$$

Thus, we obtain

$$a_j(r) = \frac{r \sinh(\sqrt{\lambda_j}/r)}{r_0 \sinh(\sqrt{\lambda_j}/r_0)} \langle \psi(r_0), \chi_j \rangle + r Q_j(r). \tag{2.183}$$

where

$$rQ_j(r) = -\frac{r\sinh(\sqrt{\lambda_j}/r)}{\sqrt{\lambda_j}\sinh(\sqrt{\lambda_j}/r_0)}\int_{r_0}^{\infty}\sinh\left(\frac{\sqrt{\lambda_j}}{r_0}-\frac{\sqrt{\lambda_j}}{s}\right)\cdot\frac{h_j(s)}{s}ds \tag{2.184}$$

$$+\frac{r}{\sqrt{\lambda_j}}\int_{r}^{\infty}\sinh\left(\frac{\sqrt{\lambda_j}}{r}-\frac{\sqrt{\lambda_j}}{s}\right)\cdot\frac{h_j(s)}{s}ds. \tag{2.185}$$

Next, we would like to expand $Q_j(r)$ in order to extract the highest order term. Multiplying the above equation by $\frac{\sqrt{\lambda_j}}{r\sinh(\sqrt{\lambda_j}/r)}$ we have

$$\frac{1}{\sinh(\sqrt{\lambda_j}/r_0)}\int_{r_0}^{\infty}\sinh\left(\frac{\sqrt{\lambda_j}}{r_0}-\frac{\sqrt{\lambda_j}}{s}\right)\cdot\frac{h_j(s)}{s}ds \tag{2.186}$$

$$=-\frac{\sqrt{\lambda_j}}{\sinh(\sqrt{\lambda_j}/r)}Q_j(r)+\frac{1}{\sinh(\sqrt{\lambda_j}/r)}\int_{r}^{\infty}\sinh\left(\frac{\sqrt{\lambda_j}}{r}-\frac{\sqrt{\lambda_j}}{s}\right)\cdot\frac{h_j(s)}{s}ds. \tag{2.187}$$

Taking the radial derivative, we obtain

$$0=-\frac{\sqrt{\lambda_j}}{\sinh(\sqrt{\lambda_j}/r)}Q_j'(r)-\frac{\sqrt{\lambda_j}}{\sinh^2(\sqrt{\lambda_j}/r)}\cdot\cosh(\sqrt{\lambda_j}/r)\cdot\frac{\sqrt{\lambda_j}}{r^2}Q_j(r) \tag{2.188}$$

$$-\frac{\sqrt{\lambda_j}}{r^2\sinh^2(\sqrt{\lambda_j}/r)}\int_{r}^{\infty}\sinh\left(\frac{\sqrt{\lambda_j}}{s}\right)\frac{h_j(s)}{s}ds, \tag{2.189}$$

and thus

$$Q_j'(r)=-\frac{\sqrt{\lambda_j}\cosh(\sqrt{\lambda_j}/r)}{r^2\sinh(\sqrt{\lambda_j}/r)}Q_j(r)\underbrace{-\frac{1}{r^2\sinh(\sqrt{\lambda_j}/r)}\int_{r}^{\infty}\sinh\left(\frac{\sqrt{\lambda_j}}{s}\right)\frac{h_j(s)}{s}ds}_{:=F_j(r)}. \tag{2.190}$$

Line (2.190) is a first order ODE for $Q_j(r)$ with boundary condition

$$Q_j(r_0) = 0, \quad \lim_{r\to\infty} Q_j(r) = 0. \tag{2.191}$$

Note that

$$-\frac{\sqrt{\lambda_j}\cosh(\sqrt{\lambda_j}/r)}{r^2\sinh(\sqrt{\lambda_j}/r)} = \frac{d}{dr}\log\sinh\left(\frac{\sqrt{\lambda_j}}{r}\right). \tag{2.192}$$

Thus, we can use the fundamental theorem of calculus to write

$$\begin{aligned} Q_j(r) =& e^{-\int_{r_0}^{r} \frac{\sqrt{\lambda_j}\cosh(\sqrt{\lambda_j}/s)}{s^2\sinh(\sqrt{\lambda_j}/s)}ds} \int_{r_0}^{r} e^{\int_{r_0}^{s} \frac{\sqrt{\lambda_j}\cosh(\sqrt{\lambda_j}/t)}{t^2\sinh(\sqrt{\lambda_j}/t)}dt} F_j(s)ds && (2.193)\\ =& \int_{r_0}^{r} \frac{\sinh(\sqrt{\lambda_j}/r)}{\sinh(\sqrt{\lambda_j}/s)} F_j(s)ds && (2.194)\\ =& -\int_{r_0}^{r} \frac{\sinh(\sqrt{\lambda_j}/r)}{\sinh(\sqrt{\lambda_j}/s)} \left[\frac{1}{s^2}\int_s^\infty \frac{h_j(t)}{t}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right] ds. && (2.195) \end{aligned}$$

Hence, we have

$$\begin{aligned} a_j(r) =& \frac{r\sinh(\sqrt{\lambda_j}/r)}{r_0\sinh(\sqrt{\lambda_j}/r_0)}\langle\psi(r_0),\chi_j\rangle && (2.196)\\ & -\int_{r_0}^{r} \frac{\sinh(\sqrt{\lambda_j}/r)}{\sinh(\sqrt{\lambda_j}/s)} \left[\frac{1}{s^2}\int_s^\infty \frac{h_j(t)}{t}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right] ds. && (2.197) \end{aligned}$$

This allows us to finally obtain our expansion by separating terms depending on their decay at infinity. We define

$$\psi = A + \tilde{\psi}, \tag{2.198}$$

where A is the leading order term

$$A = D_0\chi_0 + \sum_{j\neq 0}\left(\frac{\sqrt{\lambda_j}}{r_0\sinh(\sqrt{\lambda_j}/r_0)}\langle\psi(r_0),\chi_j\rangle + c_j\right)\chi_j \tag{2.199}$$

with

$$c_j = -\int_{r_0}^{\infty} \frac{\sqrt{\lambda_j}}{\sinh(\sqrt{\lambda_j}/s)} \left[\frac{1}{s^2}\int_s^{\infty} \frac{h_j(t)}{t}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right] ds, \tag{2.200}$$

and $\tilde{\psi}$ is the lower order term

$$\tilde{\psi} = \chi_0 \int_r^{\infty} \left[\int_s^{\infty} \frac{h_0(t)}{t^2}dt\right] ds \tag{2.201}$$

$$+ \sum_{j\neq 0} \left(\frac{r\sinh(\sqrt{\lambda_j}/r)}{r_0\sinh(\sqrt{\lambda_j}/r_0)} - \frac{\sqrt{\lambda_j}}{r_0\sinh(\sqrt{\lambda_j}/r_0)}\right)\langle\psi(r_0),\chi_j\rangle\chi_j + \gamma \tag{2.202}$$

with

$$\gamma = \sum_{j=1}^{\infty} rQ_j(r)\chi_j - \sum_{j=1}^{\infty} c_j\chi_j. \tag{2.203}$$

Even though this expansion looks somewhat intimidating, both A and $\tilde{\psi}$ actually have a very nice form. The only problematic term is γ and most of the remainder of this section focuses on controlling γ.

$$rQ_j(r) = \underbrace{-\int_{r_0}^{\infty} \frac{\sqrt{\lambda_j}}{\sinh(\sqrt{\lambda_j}/s)} \left[\frac{1}{s^2}\int_s^{\infty} \frac{\tilde{h}_j(t)}{t}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right] ds}_{:=c_j} \tag{2.204}$$

$$\underbrace{-\int_{r_0}^{r} \frac{r\sinh(\sqrt{\lambda_j}/r) - \sqrt{\lambda_j}}{\sinh(\sqrt{\lambda_j}/s)} \left[\frac{1}{s^2}\int_s^{\infty} \frac{\tilde{h}_j(t)}{t}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right] ds}_{:=E_j^{(2)}} \tag{2.205}$$

$$+\underbrace{\int_{r}^{\infty} \frac{\sqrt{\lambda_j}}{\sinh(\sqrt{\lambda_j}/s)} \left[\frac{1}{s^2}\int_s^{\infty} \frac{\tilde{h}_j(t)}{t}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right] ds}_{:=E_j^{(3)}}. \tag{2.206}$$

Regularity of ψ implies improved decay rates for h

In this section we show how estimates on h imply estimates on ψ. We show in the next section that the opposite direction also holds which allows us then to perform a bootstrap argument and prove Theorem 19. Recall that

$$\begin{aligned} h = &-\partial_r^2\psi - \frac{2}{r}\partial_r\psi + \frac{15}{4}t^{-2}\psi && (2.207)\\ &- \frac{3}{2}|v|^{\frac{1}{2}}t^{-\frac{3}{2}}\left(|\nabla\psi|^2 - \left|\frac{\nabla|v|}{|v|}\cdot\nabla\psi\right|^2\right) - 3|v|^{-1}\psi\nabla(|v|^{\frac{3}{2}}t^{-\frac{3}{2}})\cdot\nabla\psi && (2.208)\\ &+ \frac{9}{2}|v|^{-\frac{1}{2}}t^{-\frac{5}{2}}\psi\frac{\nabla|v|}{|v|}\cdot\nabla\psi + \left(\frac{27}{4}|v|^{-\frac{1}{2}}t^{-\frac{5}{2}} - \frac{27}{8}|v|^{-\frac{3}{2}}t^{-\frac{7}{2}}\right)\psi^2 && (2.209)\\ &+ O_k(|v|^{\frac{1}{2}}t^{-\frac{7}{2}}). && (2.210) \end{aligned}$$

Lemma 21. *1. We have $\int_{S^2} h^2 d\sigma = O(r^{-2})$, $\int_{S^2}|\nabla^{S^2}h|^2 d\sigma = O(1)$.*

2. If $\int_{S^2} h^2 d\sigma = O(r^{-2l})$ for some $l \geq 1$, then $\sum_{j\neq 0}\lambda_j^d\langle h,\chi_j\rangle^2 = O(r^{-2l+2d+\varepsilon})$ for any $\varepsilon > 0$.

Proof. (1) The estimate $\psi = O_2(1)$ implies $h = O_1(|v|^{\frac{1}{2}}t^{-\frac{3}{2}} + r^{-2})$ from which the result follows.

(2) From Corollary 16 we obtain $\psi = O_k(1)$ for any k. Thus, We have $h = O_1(r^{-1})$. For $k \gg 1$ to be determined later, we obtain

$$\begin{aligned} \sum_{j\neq 0}\lambda_j^d\langle h,\chi_j\rangle^2 \leq &\left(\sum_{j\neq 0}\langle h,\chi_j\rangle^2\right)^{\frac{k-d}{k}}\cdot\left(\sum_{j\neq 0}\lambda_j^k\langle h,\chi_j\rangle^2\right)^{\frac{d}{k}} && (2.211)\\ =&O(r^{-2l\cdot\frac{k-d}{k}+2d}) && (2.212) \end{aligned}$$

For any fixed l, d and $\varepsilon > 0$, we can find $k \gg 1$ such that $-2l\cdot\frac{k-d}{k}+2d \leq -2l+2d+\varepsilon$.

This finishes the proof. $\square$

Lemma 22. *1. Suppose* $\int_{S^2} |\nabla^{S^2}\psi|^2 = O(r^\alpha)$ *for some* $0 \le \alpha < 1$*. Then we have* $\int_{S^2} h^2 d\sigma = O(r^{-4+\alpha})$.

2. Let $0 \le \varepsilon < 1$*. If* ψ *satisfies the estimates*

$$\int_{S^2} |\nabla^{S^2}\psi|^2 d\sigma = O(r^\varepsilon), \qquad \int_{S^2} |\partial_r \psi|^2 d\sigma = O(r^{-4+\varepsilon}), \tag{2.213}$$

$$\int_{S^2} |\nabla^{S^2}\psi|^4 d\sigma = O(r^\varepsilon), \qquad \int_{S^2} |\partial_r^2 \psi|^2 d\sigma = O(r^{-6+\varepsilon}), \tag{2.214}$$

we obtain $\int_{S^2} |(\nabla^{S^2})^d h|^2 d\sigma = O(r^{-6+2d+\varepsilon+\varepsilon'})$ *for any* $\varepsilon' > 0$ *and* $d \ge 1$.

Proof. (1). Observe that

$$\frac{\nabla |v|}{|v|} = \left(\frac{1}{r} - \frac{1}{rt|v|}\right)\nabla r + r^{-1}\frac{\nabla^{S^2}|v|}{|v|}. \tag{2.215}$$

We compute

$$|\nabla\psi|^2 - \left|\frac{\nabla|v|}{|v|}\cdot\nabla\psi\right|^2 \tag{2.216}$$

$$= \left(\nabla\psi - \frac{\nabla|v|}{|v|}\cdot\nabla\psi\right)\cdot\left(\nabla\psi + \frac{\nabla|v|}{|v|}\cdot\nabla\psi\right) \tag{2.217}$$

$$\le 2|\nabla\psi|\cdot\left|\nabla\psi - \frac{\nabla|v|}{|v|}\cdot\nabla\psi\right| \tag{2.218}$$

$$\le 2|\nabla\psi|\cdot\left|\left[\left(\frac{\nabla r}{|\nabla r|} - \left(\frac{1}{r} - \frac{1}{rt|v|}\right)\nabla r\right)\cdot\nabla\psi\right]^2 + r^{-2}|\nabla^{S^2}\psi|^2\right|^{1/2} \tag{2.219}$$

$$\le C|\nabla\psi|\cdot\left[\left(\frac{1}{t} - \frac{1}{r} + \frac{1}{rt|v|}\right)|t^2\partial_r\psi| + r^{-1}|\nabla^{S^2}\psi|\right] \tag{2.220}$$

$$\le C|v|^{-1}\cdot|\nabla\psi|\cdot|\partial_r\psi| + Cr^{-1}|\nabla\psi|\cdot|\nabla^{S^2}\psi| \tag{2.221}$$

$$\le C|v|^{-1}r^{-1} + Cr^{-1}|\nabla^{S^2}\psi|. \tag{2.222}$$

Thus, we have

$$\int_{S^2}\left[|v|^{\frac{1}{2}}t^{-\frac{3}{2}}\left(|\nabla\psi|^2-\left|\frac{\nabla|v|}{|v|}\cdot\nabla\psi\right|^2\right)\right]^2 d\sigma \tag{2.223}$$

$$\leq C\int_{S^2}|v|^{-1}t^{-5}+|v|t^{-5}|\nabla^{S^2}\psi|^2 d\sigma \tag{2.224}$$

$$\leq O(r^{-6}\ln r)+Ct^{-4}\int_{S^2}|\nabla^{S^2}\psi|^2 d\sigma \tag{2.225}$$

$$=O(r^{-6}\ln r+r^{-4+\alpha}). \tag{2.226}$$

Since $|\nabla(|v|^{\frac{3}{2}}t^{-\frac{3}{2}})|^2=\frac{9}{2}|v|^2t^{-4}-\frac{9}{4}|v|^3t^{-5}$, we have

$$\int_{S^2}\left(|v|^{-1}\psi\nabla(|v|^{\frac{3}{2}}t^{-\frac{3}{2}})\cdot\nabla\psi\right)^2 d\sigma \leq C\int_{S^2}t^{-4}\psi^2|\nabla\psi|^2 d\sigma \tag{2.227}$$

$$=O(r^{-4+\alpha}). \tag{2.228}$$

(2) We first show

$$\int_{S^2}(h-\frac{15}{4}t^{-2}\psi)^2 d\sigma=O(r^{-6+\varepsilon}). \tag{2.229}$$

Since

$$|\nabla\psi|^2-\left|\frac{\nabla|v|}{|v|}\cdot\nabla\psi\right|^2 \tag{2.230}$$

$$=r^{-2}|\nabla^{S^2}\psi|^2+\left[\left(\frac{\nabla r}{|\nabla r|}-\frac{\nabla|v|}{|v|}\right)\cdot\nabla\psi\right]\cdot\left[\left(\frac{\nabla r}{|\nabla r|}+\frac{\nabla|v|}{|v|}\right)\cdot\nabla\psi\right], \tag{2.231}$$

and

$$\frac{\nabla r}{|\nabla r|}-\frac{\nabla|v|}{|v|}=\frac{1-\cos\theta}{t|v|}\cdot\frac{\nabla r}{|\nabla r|}+\frac{\sin\theta}{|v|}\cdot r\nabla\theta \tag{2.232}$$

$$=O(t^{-1}|v|^{-1}+t^{-\frac{1}{2}}|v|^{-\frac{1}{2}}) \tag{2.233}$$

we obtain

$$\int_{S^2}\left[|v|^{\frac{1}{2}}t^{-\frac{3}{2}}\cdot\left(|\nabla\psi|^2-\left|\frac{\nabla|v|}{|v|}\cdot\nabla\psi\right|^2\right)\right]^2 d\sigma \tag{2.234}$$

$$\leq C\int_{S^2} r^{-6}|\nabla^{S^2}\psi|^4 d\sigma + C\int_{S^2} t^{-4}|\nabla\psi|^4 d\sigma \tag{2.235}$$

$$\leq Cr^{-6+\varepsilon}+Ct^{-4}\int_{S^2}|\nabla\psi|^2 d\sigma \tag{2.236}$$

$$=O(r^{-6+\varepsilon}). \tag{2.237}$$

Since $|\nabla(|v|^{\frac{3}{2}}t^{-\frac{3}{2}})|=O(|v|t^{-2})$, $\psi=O(1)$, we have

$$\int_{S^2}[|v|^{-1}\psi\nabla(|v|^{\frac{3}{2}}t^{-\frac{3}{2}})\cdot\nabla\psi]^2 d\sigma \leq t^{-4}\int_{S^2}|\nabla\psi|^2 d\sigma \tag{2.238}$$

$$\leq t^{-2}\int_{S^2}|\partial_r\psi|^2 d\sigma + t^{-6}\int_{S^2}|\nabla^{S^2}\psi|^2 d\sigma \tag{2.239}$$

$$=O(t^{-6+\varepsilon}). \tag{2.240}$$

Since $\psi=O_k(1)$, then the last three terms in h, cf. equation 2.207, can be simplified via

$$\frac{9}{2}|v|^{-\frac{1}{2}}t^{-\frac{5}{2}}\psi\frac{\nabla|v|}{|v|}\cdot\nabla\psi+\left(\frac{27}{4}|v|^{-\frac{1}{2}}t^{-\frac{5}{2}}-\frac{27}{8}|v|^{-\frac{3}{2}}t^{-\frac{7}{2}}\right)\psi^2+O_k(|v|^{\frac{1}{2}}t^{-\frac{7}{2}}) \tag{2.241}$$

$$=O(|v|^{-\frac{1}{2}}t^{-\frac{5}{2}}). \tag{2.242}$$

Next, we make the claim

$$\int_{S^2}\frac{1}{|v|^\alpha}d\sigma=\begin{cases}O(r^{\alpha-2}) & \text{for } \alpha>1,\\ O(r^{-1}\ln r) & \text{for } \alpha=1,\\ O(r^{-\alpha}) & \text{for } 0<\alpha<1.\end{cases} \tag{2.243}$$

Using the identities $|v| = \sqrt{1+r^2} + r\cos\theta = r + \frac{1}{2}r^{-1} + O(r^{-2}) + r\cos\theta$ and $\cos\theta = -1 + \frac{(\pi-\theta)^2}{2} + O((\pi-\theta)^4)$ for θ near π, we obtain for a small, fixed $\varepsilon > 0$

$$\int_{S^2} |v|^{-\alpha} d\sigma \leq r^{-\alpha} \int_0^{2\pi} \frac{\sin\theta d\theta}{(1+\cos\theta+\frac{r^{-2}}{3})^\alpha} \tag{2.244}$$

$$= Cr^{-\alpha} + r^{-\alpha} \int_{\pi-\varepsilon}^{\pi+\varepsilon} \frac{\sin\theta d\theta}{(\frac{(\pi-\theta)^2}{2}+\frac{r^{-2}}{3})^\alpha} \tag{2.245}$$

$$\leq Cr^{-\alpha} + 9r^{-\alpha} \int_{-\varepsilon}^{\varepsilon} \frac{\theta d\theta}{(\theta^2+r^{-2})^\alpha} \tag{2.246}$$

$$= Cr^{-\alpha} + 18r^{-\alpha} \int_{r^{-1}}^{\varepsilon} \frac{\theta d\theta}{(\theta^2+r^{-2})^\alpha} + 18r^{-\alpha} \int_0^{r^{-1}} \frac{\theta d\theta}{(\theta^2+r^{-2})^\alpha} \tag{2.247}$$

$$\leq Cr^{-\alpha} + 18r^{-\alpha} \int_{r^{-1}}^{\varepsilon} \frac{d\theta}{\theta^{2\alpha-1}} + 18 \int_0^{r^{-1}} r^\alpha \theta d\theta. \tag{2.248}$$

This implies the claim. Thus, we have

$$\int_{S^2} |v|^{-1} t^{-5} d\sigma = O(t^{-6}\ln t) = O(t^{-6+\varepsilon}), \tag{2.249}$$

which yields

$$\int_{S^2} (h - \frac{15}{4}t^{-2}\psi)^2 d\sigma = O(r^{-6+\varepsilon}). \tag{2.250}$$

Since $\psi = O_k(1)$, for any $k \in \mathbb{N}$, we have (by checking term for term) $h - \frac{15}{4}t^{-2}\psi = O_k(r^{-1})$ for any $k \in \mathbb{N}$. Therefore, we have

$$\int_{S^2} |(\nabla^{S^2})^k (h - \frac{15}{4}t^{-2}\psi)|^2 d\sigma = O(r^{-2+2k}). \tag{2.251}$$

Using the Gagliardo–Nirenberg interpolation inequality, we have for $d \in (0, k)$

$$\int_{S^2} |(\nabla^{S^2})^d (h - \frac{15}{4} t^{-2} \psi)|^2 d\sigma \tag{2.252}$$

$$\leq C_{d,k} \left[\int_{S^2} (h - \frac{15}{4} t^{-2} \psi)^2 d\sigma \right]^{\frac{k-d}{k}} \cdot \left[\int_{S^2} |(\nabla^{S^2})^k (h - \frac{15}{4} t^{-2} \psi)|^2 d\sigma \right]^{\frac{d}{k}} \tag{2.253}$$

$$= O(C_{d,k} r^{\frac{(-6+\varepsilon)(k-d)}{k} + \frac{(-2+2k)d}{k}}). \tag{2.254}$$

Here $C_{d,k}$ depends on d and k. Fixing d and letting $k \to \infty$, we have $\frac{(-6+\varepsilon)(k-d)}{k} + \frac{(-2+2k)d}{k} \to -6 + 2d + \varepsilon$. Hence, we have for any $\varepsilon' > 0$

$$\int_{S^2} |(\nabla^{S^2})^d (h - \frac{15}{4} t^{-2} \psi)|^2 d\sigma = O(C_{d,\varepsilon,\varepsilon'} r^{-6+2d+\varepsilon+\varepsilon'}). \tag{2.255}$$

Since

$$\int_{S^2} |\nabla^{S^2} \psi|^2 d\sigma = O(r^{\varepsilon}) \tag{2.256}$$

and $\psi = O_k(1)$ for all $k \in \mathbb{N}$, we can proceed as above to obtain for $d \geq 1$

$$\int_{S^2} |(\nabla^{S^2})^d \psi|^2 d\sigma = O(r^{-2+2d+\varepsilon+\varepsilon'}), \tag{2.257}$$

Therefore, for $d \geq 1$

$$\int_{S^2} |(\nabla^{S^2})^d h|^2 d\sigma = O(r^{-6+2d+\varepsilon+\varepsilon'}) \tag{2.258}$$

which finishes the proof. □

Improved decay rates for h imply regularity for A

In this section we show how the regularity of A depends on the decay rates of $\tilde{h}$. Recall that

$$A = A_0\chi_0 + \sum_{j\neq 0}\left(\frac{\sqrt{\lambda_j}}{r_0\sinh(\sqrt{\lambda_j}/r_0)}\langle\psi(r_0),\chi_j\rangle + c_j\right)\chi_j. \tag{2.259}$$

where c_j are given by

$$c_j = -\int_{r_0}^{\infty}\frac{\sqrt{\lambda_j}}{\sinh(\sqrt{\lambda_j}/s)}\left[\frac{1}{s^2}\int_s^{\infty}\frac{h_j(t)}{t}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right]ds. \tag{2.260}$$

Proposition 23. *1. If $\int_{S^2} h^2 d\sigma = O(r^{-2l})$ for some $l > 0$, then $A \in H^{l-\varepsilon}(S^2)$ for any $\varepsilon > 0$.*

2. If $\int_{S^2} |(\nabla^{S^2})^d h|^2 d\sigma = O(r^{-2l})$ for some $d, l > 0$, then $A \in H^{l+d-\varepsilon}(S^2)$ for any $\varepsilon > 0$.

Proof. Our goal is to show $\sum_j \lambda_j^{l-\delta} c_j^2 < \infty$. First, we observe

$$\sum_j\left(\int_s^{\infty}\frac{h_j(t)}{t}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right)^2 = O(s^{-2l}). \tag{2.261}$$

Next, we estimate

$$\sum_j \lambda_j^{l-\delta} c_j^2 \tag{2.262}$$

$$= \sum_j \lambda_j^{l-\delta} \left[\int_{r_0}^{\infty} \frac{\sqrt{\lambda_j}}{\sinh(\sqrt{\lambda_j}/s)} \left(\frac{1}{s^2} \int_s^{\infty} \frac{h_j(t)}{t} \frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)} dt \right) ds \right]^2 \tag{2.263}$$

$$\leq \sum_j \lambda_j^{l-\delta} \int_{r_0}^{\infty} \frac{\lambda_j s^{-3-2l+2\delta}}{\sinh^2(\sqrt{\lambda_j}/s)} ds \cdot \int_{r_0}^{\infty} s^{2l-2\delta-1} \left(\int_s^{\infty} \frac{h_j(t)}{t} \frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)} dt \right)^2 ds. \tag{2.264}$$

Moreover, we have

$$\lambda_j^{l-\delta} \int_{r_0}^{\infty} \frac{\lambda_j s^{-3-2l+2\delta}}{\sinh^2(\sqrt{\lambda_j}/s)} ds = \int_{r_0}^{\infty} \frac{(\sqrt{\lambda_j}/s)^{1+2l-2\delta}}{\sinh^2(\sqrt{\lambda_j}/s)} \cdot \frac{\sqrt{\lambda_j} ds}{s^2} \tag{2.265}$$

$$= \int_0^{\frac{\sqrt{\lambda_j}}{r_0}} \frac{(\bar{s})^{1+2l-2\delta}}{\sinh^2(\bar{s})} d\bar{s} \tag{2.266}$$

$$\leq \int_0^{\infty} \frac{(\bar{s})^{1+2l-2\delta}}{\sinh^2(\bar{s})} d\bar{s} < \infty. \tag{2.267}$$

This proves the proposition. □

Improved decay rates for $\tilde{h}$ imply regularity for $\tilde{\psi}$

Next, we analyze decay rate of the error term $\tilde{\psi} = \psi - A$. Recall that

$$\tilde{\psi} = \chi_0 \tilde{\psi}_1 + \tilde{\psi}_2 + \sum_{j \neq 0} \gamma_j \chi_j \tag{2.268}$$

where

$$\tilde{\psi}_2 := \sum_{j \neq 0} \left(\frac{r \sinh(\sqrt{\lambda_j}/r)}{r_0 \sinh(\sqrt{\lambda_j}/r_0)} - \frac{\sqrt{\lambda_j}}{r_0 \sinh(\sqrt{\lambda_j}/r_0)} \right) \langle \psi(r_0), \chi_j \rangle \chi_j \tag{2.269}$$

and

$$\gamma_j = -\int_{r_0}^{r} \frac{r\sinh(\sqrt{\lambda_j}/r) - \sqrt{\lambda_j}}{\sinh(\sqrt{\lambda_j}/s)} \left[\frac{1}{s^2}\int_s^\infty \frac{h_j(t)}{t}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right] ds \tag{2.270}$$

$$+\int_r^\infty \frac{\sqrt{\lambda_j}}{\sinh(\sqrt{\lambda_j}/s)} \left[\frac{1}{s^2}\int_s^\infty \frac{h_j(t)}{t}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right] ds \tag{2.271}$$

and

$$\tilde{\psi}_1 = \int_r^\infty \int_s^\infty \frac{h_0(t)}{t^2} dt ds \tag{2.272}$$

To show regularity for $\tilde{\psi}$ we have to control all of the above terms. We begin with the first and simplest one:

Proposition 24. *If $\int_{S^2} h^2(r) d\sigma = O(r^{-2l})$, for some $l > 0$, then $\tilde{\psi}_1$ satisfies*

$$\tilde{\psi}_1(r) = O(r^{-l}),\ \tilde{\psi}_1'(r) = O(r^{-l-1}),\ \tilde{\psi}_1''(r) = O(r^{-l-2}). \tag{2.273}$$

Proof. $\int_{S^2} h^2(r) d\sigma = O(r^{-2l})$ implies $h_0(r) = O(r^{-l})$, by Cauchy-Schwarz inequality. Therefore, $\tilde{\psi}_1''(r) = r^{-2} h_0(r) = O(r^{-l-2})$, then integrating $\tilde{\psi}_1''(r)$, we have

$$\tilde{\psi}_1'(r) = O(r^{-l-1}) \text{ and } \tilde{\psi}_1(r) = O(r^{-l}) \tag{2.274}$$

which finishes the proof. □

Next, we estimate $\tilde{\psi}_2$:

Proposition 25. *For any $d \geq 0$ and non-negative integer k, we have*

$$\sum_{j\neq 0} \lambda_j^d \langle \partial_r^k \tilde{\psi}_2, \chi_j\rangle^2 = O(r^{-4-2k}) \tag{2.275}$$

Proof. For $r \geq r_0$, $r_0 \gg 1$, we have

$$\begin{aligned}
&\left|\partial_r^k \left[r\sinh(\sqrt{\lambda_j}/r) - \sqrt{\lambda_j}\right]\right| && (2.276)\\
=&\left|\partial_r^k \left[\sum_{i=1}^{\infty} \frac{1}{(2i+1)!}\lambda_j^{\frac{2i+1}{2}} r^{-2i}\right]\right| && (2.277)\\
=&\left|\sum_{i=1}^{\infty} \frac{\prod_{q=1}^{k}(2i-1+q)}{(2i+1)!}\lambda_j^{\frac{2i+1}{2}} \cdot (-1)^k r^{-2i-k}\right| && (2.278)\\
\leq&\sum_{i=1}^{k+1} \frac{\prod_{q=1}^{k}(2i-1+q)}{(2i+1)!}\lambda_j^{\frac{2i+1}{2}} r^{-2i-k} + \sum_{i=k+2}^{\infty} \frac{2^k}{(2i+1-k)!}\lambda_j^{\frac{2i+1}{2}} r^{-2i-k} && (2.279)\\
\leq&\sum_{i=1}^{k+1} \frac{\prod_{q=1}^{k}(2i-1+q)}{(2i+1)!}\lambda_j^{\frac{2i+1}{2}} r^{-2i-k} + 2^k \lambda_j^{k+2} r^{-3k-4}\sinh(\lambda_j^{\frac{1}{2}} r^{-1}) && (2.280)\\
\leq& C_k(\lambda_j^{\frac{3}{2}} r^{-2-k} + \lambda_j^{\frac{2k+3}{2}} r^{-3k-2}) + 2^k \lambda_j^{k+2} r^{-3k-4}\sinh(\lambda_j^{\frac{1}{2}} r_0^{-1}), && (2.281)
\end{aligned}$$

where C_k is a constant depending on k, and we set $\prod_{q=1}^{k}(2i-1+q) = 1$ for $k = 0$.

Next, we compute

$$\begin{aligned}
&\sum_{j\neq 0} \lambda_j^d \langle \partial_r^k \tilde{\psi}_2, \chi_j\rangle^2 && (2.282)\\
=&\sum_{j\neq 0} \left|\partial_r^k \left[r\sinh(\lambda_j^{\frac{1}{2}} r^{-1}) - \lambda_j^{\frac{1}{2}}\right]\right|^2 [r_0 \sinh(\lambda_j^{\frac{1}{2}} r_0^{-1})]^{-2} \cdot \lambda_j^d \langle \psi(r_0), \chi_j\rangle^2 && (2.283)\\
\leq&\sum_{j\neq 0} \left[C_k(\lambda_j^{\frac{3}{2}} r^{-2-k} + \lambda_j^{\frac{2k+3}{2}} r^{-3k-2}) + 2^k \lambda_j^{k+2} r^{-3k-4}\sinh(\lambda_j^{\frac{1}{2}} r_0^{-1})\right]^2 && (2.284)\\
&[r_0 \sinh(\lambda_j^{\frac{1}{2}} r_0^{-1})]^{-2} \lambda_j^d \langle \psi(r_0), \chi_j\rangle^2 && (2.285)\\
\leq&\sum_{j\neq 0} \tilde{C}_k(\lambda_j^2 r^{-4-2k} + \lambda_j^{2k+2} r^{-6k-4} + \lambda_j^{2k+4} r^{-6k-8} r_0^{-2})\lambda_j^d \langle \psi(r_0), \chi_j\rangle^2 && (2.286)
\end{aligned}$$

where in line 2.286 we applied the estimate $r_0 \sinh(\lambda_j^{\frac{1}{2}} r_0^{-1}) \geq \lambda_j^{\frac{1}{2}}$, and $\tilde{C}_k$ is a large constant only depending on k. Since $\psi(r)$ is smooth, we have $\sum_{j\neq 0} \lambda_j^d \langle \psi(r_0), \chi_j\rangle^2 < \infty$ for any $d \geq 0$. Therefore, we can absorb arbitrary powers of λ_j into $\langle \psi(r_0), \chi_j\rangle^2$

which leads to

$$\sum_{j\neq 0} \lambda_j^d \langle \partial_r^k \tilde{\psi}_2, \chi_j\rangle^2 = O(r^{-4-2k}) \tag{2.287}$$

as desired. ☐

It remains to estimate $\gamma = \sum_j \gamma_j \chi_j$. For this purpose, we first show a technical lemma:

Lemma 26. *Suppose that $\sum_j h_j^2(r) = O(r^{-2l})$ for some $l > -1$. Then we have*

$$\sum_j \left(\int_s^\infty \frac{h_j(t)}{t} \frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)} dt \right)^2 = O(s^{-2l}). \tag{2.288}$$

Proof. We compute

$$\begin{aligned}
\sum_j \left(\int_s^\infty \frac{h_j(t)}{t} \frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)} dt \right)^2 &\leq \sum_j \left(\int_s^\infty h_j(t) s dt \right)^2 &(2.289)\\
&\leq \sum_j s^2 \int_s^\infty h_j^2 t^{1+\delta} dt \cdot \int_s^\infty t^{-1-\delta} dt &(2.290)\\
&= s^2 \cdot \delta^{-1} s^{-\delta} \int_s^\infty t^{1+\delta} \sum_j \tilde{h}_j^2 dt &(2.291)\\
&= O(s^{-2l}). &(2.292)
\end{aligned}$$

This finishes the proof. ☐

Using the above lemma we are now able to obtain C^0 estimates $\gamma, \partial_r\gamma$ and $\partial_r^2\gamma$. Since the proof is rather lengthy, we split it up into three proposition.

Proposition 27. *Suppose that $\sum_j \lambda_j^d h_j^2 = O(r^{-2l})$ for some $d \geq 0$, $l > 0$. Then we* *have*

$$\sum_{j\neq 0} \lambda_j^d \gamma_j^2 = O(r^{-2l} + r^{-4+\delta}). \tag{2.293}$$

Proof. Recall that

$$\gamma_j = -\int_{r_0}^{r} \frac{r\sinh(\sqrt{\lambda_j}/r) - \sqrt{\lambda_j}}{\sinh(\sqrt{\lambda_j}/s)} \left[\frac{1}{s^2}\int_s^\infty \frac{h_j(t)}{t}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right] ds \tag{2.294}$$

$$+\int_{r}^{\infty} \frac{\sqrt{\lambda_j}}{\sinh(\sqrt{\lambda_j}/s)} \left[\frac{1}{s^2}\int_s^\infty \frac{h_j(t)}{t}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right] ds. \tag{2.295}$$

We denote by $E_j^{(1)}$ the term in the first line and by $E_j^{(2)}$ the term in the second line. Due to Cauchy-Schwarz, it suffices to estimate $(E_j^{(1)})^2$ and $(E_j^{(2)})^2$. According to Lemma 26 and our assumption, we have

$$\sum_j \left(\int_s^\infty \lambda_j^{\frac{d}{2}} \frac{h_j(t)}{t}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right)^2 = O(s^{-2l}) \tag{2.296}$$

For $r \geq s > 0$ we have

$$\frac{r\sinh(\sqrt{\lambda_j}/r) - \sqrt{\lambda_j}}{\sinh(\sqrt{\lambda_j}/s)} \leq \frac{s^3}{r^2}. \tag{2.297}$$

This implies

$$\sum_{j=1}^{\infty} \lambda_j^d (E_j^{(2)})^2 \tag{2.298}$$

$$\leq \sum_{j=1}^{\infty} \int_{r_0}^{r} \left(\frac{s}{r^2}\right)^2 s^{1-2l-\delta} ds \cdot \int_{r_0}^{r} s^{-1+2l+\delta} \left(\int_s^\infty \lambda_j^{\frac{d}{2}} \frac{\tilde{h}_j}{t}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right)^2 ds \tag{2.299}$$

$$\leq C r^{-4} |r^{4-2l-\delta} - r_0^{4-2l-\delta}| \cdot |r^\delta - r_0^\delta| \tag{2.300}$$

$$= O(r^{-2l} + r^{-4+\delta}). \tag{2.301}$$

Using the estimate $\sinh(\sqrt{\lambda_j}/s) \geq \frac{\sqrt{\lambda_j}}{s}$, we obtain

$$\sum_{j=1}^{\infty} \lambda_j^d (E_j^{(3)})^2 \tag{2.302}$$

$$\leq \sum_{j=1}^{\infty} \int_r^\infty s^{-1+2l-\delta} \left(\int_s^\infty \lambda_j^{\frac{d}{2}} \frac{h_j(t)}{t} \frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)} dt \right)^2 ds \cdot \int_r^\infty s^{-1-2l+\delta} ds \tag{2.303}$$

$$\leq C r^{-\delta} \cdot r^{-2l+\delta} \tag{2.304}$$

$$= O(r^{-2l}) \tag{2.305}$$

which finishes the proof. □

Proposition 28. *Suppose* $\sum_j \lambda_j^d h_j^2(r) = O(r^{-2l})$ *for some* $d \geq 1$ *and* $l > -1$. *Moreover, assume that* $\sum_j \lambda_j^{d-1} h_j^2(r) = O(r^{-2-2l_1})$ *for some* $l_1 > -1$. *Then we have for any* $\varepsilon > 0$

$$\sum_{j \neq 0} \lambda_j^{d-1} (\partial_r \gamma_j)^2 = O(r^{-6} + r^{-4-2l+\varepsilon} + r^{-2-2l_1}). \tag{2.306}$$

Proof. Computing the radial derivative of γ_j, we obtain

$$\partial_r \gamma_j \tag{2.307}$$

$$= -\underbrace{\frac{1}{r} \int_r^\infty \frac{h_j(t)}{t} \frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/r)} dt}_{\tilde{E}_j^{(1)}} \tag{2.308}$$

$$+ \underbrace{\int_{r_0}^r \frac{\frac{\sqrt{\lambda_j}}{r} \cosh(\sqrt{\lambda_j}/r) - \sinh(\sqrt{\lambda_j}/r)}{\sinh(\sqrt{\lambda_j}/s)} \cdot \frac{1}{s^2} \left[\int_s^\infty \frac{h_j(t)}{t} \frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)} dt \right] ds}_{\tilde{E}_j^{(2)}} . \tag{2.309}$$

According to Lemma 26, and the identity

$$\sum_{j=1}^{\infty} \lambda_j^{d-1} h_j^2(r) = O(r^{-2l_1}), \tag{2.310}$$

we have

$$\sum_{j=1}^{\infty} \lambda_j^{d-1} (\tilde{E}_j^{(1)})^2 = O(r^{-2-2l_1}). \tag{2.311}$$

Moreover, for $0 < s \leq r$ we estimate

$$\frac{\frac{\sqrt{\lambda_j}}{r}\cosh(\sqrt{\lambda_j}/r) - \sinh(\sqrt{\lambda_j}/r)}{\sinh(\sqrt{\lambda_j}/s)} = \frac{\sum_{k=1}^{\infty} \frac{2k}{(2k+1)!}\left(\frac{\sqrt{\lambda_j}}{r}\right)^{2k+1}}{\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\left(\frac{\sqrt{\lambda_j}}{s}\right)^{2k+1}} \tag{2.312}$$

$$\leq \frac{\sum_{k=1}^{\infty} \frac{2k}{(2k+1)!}\left(\frac{\sqrt{\lambda_j}}{r}\right)^{2k+1}}{\sum_{k=1}^{\infty} \frac{1}{\sqrt{(2k-1)!\cdot(2k+1)!}}\left(\frac{\sqrt{\lambda_j}}{s}\right)^{2k}} \tag{2.313}$$

$$\leq \frac{s^2}{r^2}\cdot\frac{\sqrt{\lambda_j}}{r}. \tag{2.314}$$

Finally, we compute for any $\varepsilon > 0$

$$\sum_{j=1}^{\infty} \lambda_j^{d-1} (\tilde{E}_j^{(2)})^2 \tag{2.315}$$

$$\leq \sum_{j=1}^{\infty} \left[\int_{r_0}^{r} \frac{\lambda_j^{\frac{d}{2}}}{r^3}\left[\int_s^{\infty} t^{-1}|h_j(t)|\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right] ds\right]^2 \tag{2.316}$$

$$\leq \sum_{j=1}^{\infty} r^{-6}\int_{r_0}^{r} r^{-\varepsilon-1}ds\cdot\int_{r_0}^{r} s^{1+\varepsilon}\left[\int_s^{\infty}\lambda_j^{\frac{d}{2}} t^{-1}|h_j(t)|\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right]^2 ds \tag{2.317}$$

$$\leq \varepsilon^{-1} r^{-6}(r_0^{-\varepsilon} - r^{-\varepsilon})\sum_{j=1}^{\infty}\int_{r_0}^{r} s^{1+\varepsilon}\left[\int_s^{\infty}\lambda_j^{\frac{d}{2}}|h_j(t)|t^{-1}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right]^2 ds \tag{2.318}$$

$$\leq C r^{-6}\cdot(r_0^{-\varepsilon} - r^{-\varepsilon})\cdot\int_{r_0}^{r} s^{1-2l+\varepsilon}ds. \tag{2.319}$$

The last term equals $O(r^{-4-2l+\varepsilon}+r^{-6})$ for $l\le 1$ and $O(r^{-6})$ for $l>1$. $\square$

Proposition 29. *Suppose that* $\sum_j \lambda_j^d h_j^2(r) = O(r^{-2l})$ *for some* $d\ge 2$, $l>-1$, $\sum_j \lambda_j^{d-1} h_j^2(r) = O(r^{-2l_1})$ *for some* $l_1>-1$ *and* $\sum_j \lambda_j^{d-2} h_j^2(r) = O(r^{-2l_2})$ *for some* $l_2>-1$. *Then we have for any* $\varepsilon>0$

$$\sum_{j\neq 0}\lambda_j^{d-2}(\partial_r^2\gamma_j)^2 = O(r^{-8-2l+\varepsilon}+r^{-8}+r^{-6-2l_1}+r^{-4-2l_2}). \tag{2.320}$$

Proof. We compute

$$\partial_r^2\gamma_j = \frac{1}{r^2}\int_r^\infty \frac{h_j(t)}{t}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/r)}dt + \frac{1}{r}h_j(r) \tag{2.321}$$

$$+\frac{\frac{\sqrt{\lambda_j}}{r}\cosh(\sqrt{\lambda_j}/r)-\sinh(\sqrt{\lambda_j}/r)}{\sinh(\sqrt{\lambda_j}/r)}\cdot\frac{1}{r^2}\int_r^\infty \frac{h_j(t)}{t}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/r)}dt \tag{2.322}$$

$$\underbrace{-\frac{\lambda_j}{r^3}\sinh\left(\frac{\sqrt{\lambda_j}}{r}\right)\int_{r_0}^r \frac{1}{\sinh(\sqrt{\lambda_j}/s)}\cdot\frac{1}{s^2}\left[\int_s^\infty \frac{h_j(t)}{t}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right]ds}_{\hat{E}_j^{(1)}} \tag{2.323}$$

To estimate lines (2.321) and (2.322) we proceed as in the proof Proposition 28 and obtain that these lines decay as $O(r^{-8}+r^{-6-2l_1}+r^{-4-2l_2})$. To estimate $\hat{E}_j^{(1)}$, we begin with observing that for $0<s\le r$

$$\frac{\sinh(\sqrt{\lambda_j}/r)}{\sinh(\sqrt{\lambda_j}/s)}\le\frac{s}{r}. \tag{2.324}$$

Hence, we can estimate

$$\sum_{j=1}^{\infty}\lambda_j^{d-2}(\hat{E}_j^{(1)})^2 \tag{2.325}$$

$$\leq\sum_{j=1}^{\infty}r^{-8}\left[\int_{r_0}^{r}\frac{1}{s}\left[\int_{s}^{\infty}\lambda_j^{\frac{d}{2}}|h_j(t)|t^{-1}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right]ds\right]^2 \tag{2.326}$$

$$\leq\sum_{j=1}^{\infty}r^{-8}\int_{r_0}^{r}s^{-1-\varepsilon}dt\cdot\int_{r_0}^{r}s^{-1+\varepsilon}\left[\int_{s}^{\infty}\lambda_j^{\frac{d}{2}}|h_j(t)|t^{-1}\frac{\sinh(\sqrt{\lambda_j}/t)}{\sinh(\sqrt{\lambda_j}/s)}dt\right]^2ds \tag{2.327}$$

$$\leq Cr^{-8}(r_0^{-\varepsilon}-r^{-\varepsilon})\cdot\int_{r_0}^{r}s^{-1-2l+\varepsilon}ds \tag{2.328}$$

$$=O(r^{-8-2l+\varepsilon}+r^{-8}). \tag{2.329}$$

This completes the proof. □

Completing the bootstrap argument

Finally, we are able to establish our regularity result, Theorem 19, for A and $\tilde{\psi}$. For the convenience of the reader we recall the precise statement here:

Theorem 30. *For any $\varepsilon > 0$ the following is true:*

1. $A\in H^{3-\varepsilon}(S^2)$.
2. $\sum_{j=1}^{\infty}\tilde{\psi}_j^2=O(r^{-4+\varepsilon})$ *and* $\sum_{j=1}^{\infty}\lambda_j^d\tilde{\psi}_j^2=O(r^{-6+2d+\varepsilon})$ *for* $1\leq d\leq 3-\varepsilon$.
3. $\sum_{j=1}^{\infty}(\partial_r\tilde{\psi}_j)^2=O(r^{-6+\varepsilon})$ *and* $\sum_{j=1}^{\infty}\lambda_j^d(\partial_r\tilde{\psi}_j)^2=O(r^{-8+2d+\varepsilon})$ *for* $1\leq d\leq 3-\varepsilon$.
4. $\sum_{j=1}^{\infty}(\partial_r^2\tilde{\psi}_j)^2=O(r^{-8+\varepsilon})$ *and* $\sum_{j=1}^{\infty}\lambda_j^d(\partial_r^2\tilde{\psi}_j)^2=O(r^{-10+2d+2\varepsilon})$ *for* $1\leq d\leq 2-\varepsilon$.

Proof. Recall that

$$\tilde{\psi}=\tilde{\psi}_1\chi_0+\tilde{\psi}_2+\sum_{j\neq 0}\gamma_j\chi_j. \tag{2.330}$$

The first two terms are controlled by Proposition 24 and 25. For the γ_j we use a bootstrap argument: First, Lemma 22 shows $\int_{S^2} \tilde{h}^2 d\sigma = O(r^{-2})$, then Proposition 23 implies $A \in H^{1-\varepsilon}(S^2)$, and Proposition 27 implies $\sum_{j\neq 0} \lambda_j^{1-\varepsilon}\tilde{\psi}_j^2 = O(r^{-\varepsilon})$, where ε is a small positive number. Therefore, we have

$$\sum_{j=1}^{\infty} \lambda_j^{1-\varepsilon}\psi_j^2 = O(1). \tag{2.331}$$

Since $\psi = O_2(1)$, we obtain $\sum_{j=1}^{\infty} \lambda_j^2 \psi^2 = O(r^4)$, and thus

$$\int_{S^2} |\nabla^{S^2}\psi|^2 d\sigma \leq \left(\int_{S^2} \sum_{j=1}^{\infty} \lambda_j^{1-\varepsilon}\psi_j^2 d\sigma\right)^{\frac{1}{1+\varepsilon}} \left(\int_{S^2} \sum_{j=1}^{\infty} \lambda_j^2 \psi_j^2 d\sigma\right)^{\frac{\varepsilon}{1+\varepsilon}} = O(r^{\frac{4\varepsilon}{1+\varepsilon}}). \tag{2.332}$$

From Lemma 21, we obtain

$$\int_{S^2} h^2 d\sigma = O(r^{-4+\varepsilon}), \tag{2.333}$$

where ε is a small positive number. It is different from the one in line (2.332). By Proposition 23, we have

$$A \in H^{2-\varepsilon}(S^2). \tag{2.334}$$

Moreover, Lemma 22 gives us higher order tangential derivatives for h. Thus, we can apply Proposition 27, 28 and 29, to obtain for any $0 < \varepsilon < 1$

$$\sum_{j\neq 0} \lambda_j^d \gamma_j^2 = O(r^{-4+2d+\varepsilon}) \quad \text{for } 0 \leq d < 2-\varepsilon, \tag{2.335}$$

$$\sum_{j\neq 0} \lambda_j^d (\partial_r \gamma_j)^2 = O(r^{-6+2d+\varepsilon}) \quad \text{for } 0 \leq d < 2-\varepsilon, \tag{2.336}$$

$$\sum_{j\neq 0} \lambda_j^d (\partial_r^2 \gamma_j)^2 = O(r^{-8+2d+\varepsilon}) \quad \text{for } 0 \leq d < 1-\varepsilon. \tag{2.337}$$

Hence, we obtain $\int_{S^2} |h|^2 d\sigma = O(r^{-4})$ and $\int_{S^2} |\nabla^{S^2}\psi|^4 d\sigma = O(1)$, Moreover, Lemma 22 (4) yields $\int_{S^2} |(\nabla^{S^2})^d h|^2 d\sigma = O(r^{-6+2d+\varepsilon})$, $d \geq 1$. Finally, applying Proposition 23, 27, 28 and 29 again, we get the final estimates

$$A \in H^{3-\varepsilon}(S^2) \tag{2.338}$$

and

$$\sum_{j\neq 0} \gamma_j^2 = O(r^{-4+\varepsilon}), \quad \sum_{j\neq 0} \lambda_j^d \gamma_j^2 = O(r^{-6+2d+\varepsilon}) \quad \text{for } 1 \leq d < 3-\varepsilon, \tag{2.339}$$

$$\sum_{j} (\partial_r \gamma_j)^2 = O(r^{-6+\varepsilon}), \quad \sum_{j\neq 0} \lambda_j^d (\partial_r \gamma_j)^2 = O(r^{-8+2d+\varepsilon}) \quad \text{for } 1 \leq d \leq 3-\varepsilon, \tag{2.340}$$

$$\sum_{j\neq 0} (\partial_r^2 \gamma_j)^2 = O(r^{-8+\varepsilon}), \quad \sum_{j\neq 0} \lambda_j^d (\partial_r^2 \gamma_j)^2 = O(r^{-10+2d+2\varepsilon}) \quad \text{for } 1 \leq d \leq 2-\varepsilon. \tag{2.341}$$

This finishes the proof in view of Proposition 24 and 25. □

After we establish the decay rate of A and $\tilde{\psi}$, we can improve the C^2 estimate of ψ which is used in the boundary integral computation.

Corollary 31. *For any $\varepsilon > 0$ we have*

1. $|\nabla\psi| = O(r^{-1})$ *and* $|\nabla^2\psi| = O(r^{-2+\varepsilon})$,

2. $|\nabla w| = O(|v|r^{-2})$ *and* $|\nabla^2 w| = O(|v|r^{-2})$.

Proof. From Theorem 19 we know that

$$\sum_{j\neq 0} \lambda_j^d (\partial_r \tilde{\psi}_j)^2 = O(r^{-8+2d+\varepsilon}) \tag{2.342}$$

for all $1 \leq d \leq 3-\varepsilon$. Setting $d = 1+\varepsilon$ and using the Sobolev embedding $C^0(S^2) \subset H^{1+\varepsilon}(S^2)$, we obtain $|\partial_r \tilde{\psi}| = O(r^{-3+\frac{3}{2}\varepsilon})$. Therefore, we have $\partial_r \psi = O(r^{-2})$. Since $A \in H^{3-\varepsilon}(S^2)$ by Theorem 19, we obtain that $|\nabla^{S^2} A|$ is bounded. Again, by Theorem 19, we know that $\sum_{j\neq 0} \tilde{\psi}_j^d \gamma_j^2 = O(r^{-6+2d+\varepsilon})$ for any $1 \leq d < 3-\varepsilon$. This implies $|\nabla^{S^2} \tilde{\psi}| = O(r^{-1})$, and thus $|\nabla^{S^2} \psi| = O(1)$. Combining with the above radial derivative estimate, we have $|\nabla \psi| = O(r^{-1})$. Next, we estimate

$$|\nabla^2 \psi| \leq C \left(r^2 |\partial_r^2 \psi| + r|\partial_r \psi| + |\nabla^{S^2} \partial_r \psi| + r^{-2} |(\nabla^{S^2})^2 \psi| \right). \tag{2.343}$$

According to Theorem 19, we have $\sum_{j\neq 0} \lambda_j^d (\partial_r^2 \tilde{\psi}_j)^2 = O(r^{-10+2d+2\varepsilon})$, for $1 \leq d \leq 2-\varepsilon$, and $\sum_{j\neq 0} (\partial_r^2 \tilde{\psi}_j)^2 = O(r^{-8+\varepsilon})$. Thus, by the Sobolev imbedding $C^0(S^2) \subset H^{1+\varepsilon}(S^2)$, we have $|\partial_r^2 \tilde{\psi}| = O(r^{-4+\frac{\varepsilon}{2}})$, for any $\varepsilon > 0$. The above estimates for $\partial_r^2 \gamma$ and $\partial_r \gamma$ yield $r^2 |\partial_r^2 \psi| = O(r^{-2+\frac{\varepsilon}{2}})$, $r|\partial_r \psi| = O(r^{-2+\varepsilon})$ and $|\nabla^{S^2} \partial_r \psi| = O(r^{-2+\varepsilon})$. By Theorem 19 part (2), we have

$$\sum_j \lambda_j^{3-\varepsilon} \psi_j^2 \leq C \sum_j \lambda_j^{3-\varepsilon} (A_j^2 + \tilde{\psi}_j^2) = O(1). \tag{2.344}$$

Having C^k estimates for ψ: $\psi = O_k(1)$, for any $k \in \mathbb{N}$, leads to an $H^{3+\varepsilon}(S^2)$ estimate of ψ. More precisely,

$$\sum_j \lambda_j^{3+\varepsilon}\psi_j^2 \le \left(\sum_j \lambda_j^{3-\varepsilon}\psi_j^2\right)^{\frac{1-\varepsilon}{1+\varepsilon}} \left(\sum_j \lambda_j^4\psi_j^2\right)^{\frac{2\varepsilon}{1+\varepsilon}} = O(r^{\frac{16\varepsilon}{1+\varepsilon}}). \tag{2.345}$$

Applying the Sobolev embedding $C^2(S^2) \subset H^{3+\varepsilon}(S^2)$ yields

$$|(\nabla^{S^2})^2\psi| \le C\left(\sum_{j\neq 0} \lambda_j^{3+\varepsilon}\psi_j^2\right)^{\frac{1}{2}} + C\left(\sum_j \psi_j^2\right)^{\frac{1}{2}} = O(r^{\frac{8\varepsilon}{1+\varepsilon}}) \tag{2.346}$$

for a potentially different, $\varepsilon > 0$. Hence, for any $\varepsilon > 0$ we have the C^0 estimate $|\nabla^2\psi| = O(r^{-2+\varepsilon})$.

(2) Since $w = |v|^{\frac{3}{2}}r^{-\frac{3}{2}}\psi$, we need to estimate $\nabla(|v|^{\frac{3}{2}}r^{-\frac{3}{2}})$ and $\nabla^2(|v|^{\frac{3}{2}}r^{-\frac{3}{2}})$ to obtain the estimates for $|\nabla w|$ and $|\nabla^2 w|$. Since $|\nabla(|v|^{\frac{3}{2}}t^{-\frac{3}{2}})|^2 = \frac{9}{2}|v|^2t^{-4} - \frac{9}{4}|v|^3t^{-5}$, we have $|\nabla(|v|^{\frac{3}{2}}r^{-\frac{3}{2}})| = O(|v|t^{-2})$. Therefore,

$$|\nabla w| \le |\nabla(|v|^{\frac{3}{2}}r^{-\frac{3}{2}})|\cdot|\psi| + |\nabla\psi|\cdot|v|^{\frac{3}{2}}r^{-\frac{3}{2}} = O(|v|r^{-2}). \tag{2.347}$$

Next, we estimate

$$\nabla^2(|v|^{\frac{3}{2}}t^{-\frac{3}{2}}) \tag{2.348}$$

$$= -\frac{9}{4}|v|^{\frac{1}{2}}t^{-\frac{5}{2}}(\nabla|v|\otimes\nabla t + \nabla t\otimes\nabla|v|) + \frac{3}{4}|v|^{-\frac{1}{2}}t^{-\frac{3}{2}}\nabla|v|\otimes\nabla|v| + \frac{15}{4}|v|^{\frac{3}{2}}t^{-\frac{7}{2}}\nabla t\otimes\nabla t \tag{2.349}$$

$$+ \frac{3}{2}|v|^{\frac{1}{2}}t^{-\frac{3}{2}}\nabla^2|v| - \frac{3}{2}|v|^{\frac{3}{2}}t^{-\frac{5}{2}}\nabla^2 t \tag{2.350}$$

$$=|v|^{\frac{3}{2}}t^{-\frac{3}{2}}\left[\frac{3}{4}\frac{\nabla|v|}{|v|}\otimes\frac{\nabla|v|}{|v|} + \frac{15}{4}\frac{\nabla t}{t}\otimes\frac{\nabla t}{t} - \frac{9}{4}\left(\frac{\nabla|v|}{|v|}\otimes\frac{\nabla t}{t} + \frac{\nabla t}{t}\otimes\frac{\nabla|v|}{|v|}\right)\right] \tag{2.351}$$

$$=O(|v|t^{-2}), \tag{2.352}$$

where the last inequality is followed by $\left||v|^{-1}\nabla|v| - t^{-1}\nabla t\right|^2 = 2|v|^{-1}t^{-1}$. Therefore, we have $|\nabla^2 w| = O(|v|r^{-2})$. □

2.4.4 The higher order expansion in the purely hyperbolic region

Having established the initial expansion, we would like to further expand $\psi = A+\tilde{\psi}$. For this purpose we write

$$\tilde{\psi} = \frac{B}{r^2} + \hat{\psi} \tag{2.353}$$

for some function B on S^2. To motivate a choice for B we look at the leading order terms of the expansion. We have

$$(1+r^2)\partial_r^2\psi + \frac{2}{r}\partial_r\psi + r^{-2}\Delta_{S^2}\psi - \frac{15}{4r^2}\psi \tag{2.354}$$

$$=(1+r^2)\partial_r^2\tilde{\psi} + \frac{2}{r}\partial_r\tilde{\psi} + r^{-2}\Delta_{S^2}\psi - \frac{15}{4r^2}\tilde{\psi} + r^{-2}\Delta_{S^2}A - \frac{15}{4r^2}A \tag{2.355}$$

$$=(1+r^2)\partial_r^2\hat{\psi} + \frac{2}{r}\partial_r\hat{\psi} + r^{-2}\Delta_{S^2}\hat{\psi} + (\frac{6}{r^2} - \frac{7}{4r^4})B \tag{2.356}$$

$$+ r^{-4}\Delta_{S^2}B + r^{-2}\Delta_{S^2}A - \frac{15}{4r^2}A. \tag{2.357}$$

To cancel out the last two terms above, we set

$$B := \frac{1}{6}\left(\frac{15}{4}A - \Delta_{S^2}A\right). \tag{2.358}$$

Proposition 32. *For any $\varepsilon > 0$, we have*

$$\sum_{j=0}^{\infty}\langle\hat{\psi}, \chi_j\rangle^2 = O(r^{-6+2\varepsilon}). \tag{2.359}$$

Moreover, we have the estimates

$$\sum_{j=1}^{\infty} \lambda_j^{-1} \langle \partial_r \hat{\psi}, \chi_j \rangle^2 = O(r^{-8+2\varepsilon}) \tag{2.360}$$

and

$$\sum_{j=1}^{\infty} \lambda_j^{-2} \langle \partial_r^2 \hat{\psi}, \chi_j \rangle^2 = O(r^{-10+2\varepsilon}). \tag{2.361}$$

Proof. Define

$$S_j[f(t), a, b] = \int_a^b \frac{1}{\sinh(\sqrt{\lambda_j}/s)} \left[\frac{1}{s^2} \int_s^\infty \frac{f(t)\sinh(\sqrt{\lambda_j}/t)}{t\sinh(\sqrt{\lambda_j}/s)} dt \right] ds. \tag{2.362}$$

Then we obtain

$$\hat{\psi} = \tilde{\psi} - \frac{B}{r^2} \tag{2.363}$$

$$= \tilde{\psi} - \frac{1}{6r^2}\left(\frac{15}{4}A - \Delta_{S^2} A\right) \tag{2.364}$$

$$= \chi_0 \int_r^\infty \left[\int_s^\infty \frac{h_0(t) - \frac{15A_0}{4t^2}}{t^2} dt \right] ds + \sum_{j\neq 0} \left(\frac{r\sinh(\sqrt{\lambda_j}/r)}{r_0 \sinh(\sqrt{\lambda_j}/r_0)} - \frac{\lambda_j^{\frac{1}{2}} + \frac{\lambda_j^{\frac{3}{2}}}{6r^2}}{r_0\sinh(\sqrt{\lambda_j}/r_0)} \right) \langle \psi(r_0), \chi_j \rangle \chi_j \tag{2.365}$$

$$+ \sum_{j=1}^{\infty} \chi_j \Bigg[-r\sinh(\sqrt{\lambda_j}/r) S_j[h_j(t), r_0, r] + (\lambda_j^{\frac{1}{2}} + \frac{\lambda_j^{\frac{3}{2}}}{6r^2}) S_j[h_j(t), r_0, \infty] - \frac{15}{24r^2} A_j \Bigg]. \tag{2.366}$$

We denote the two terms in line (2.365) by $\hat{\psi}_1$ and $\hat{\psi}_2$. Since $h_0(t) - \frac{15A_0}{4t^2} = O(t^{-3+\varepsilon})$, we obtain similar to Proposition 24 $\hat{\psi}_1(r) = O(r^{-3+\varepsilon})$, $\hat{\psi}_1'(r) = O(r^{-4+\varepsilon})$ and $\hat{\psi}_1''(r) = O(r^{-5+\varepsilon})$. Similar to Proposition 25, we have for any $k \in \mathbb{N}$

$$\sum_{j\neq 0} \langle \partial_r^k \hat{\psi}_2(r), \chi_j \rangle^2 = O(r^{-2k-8}). \tag{2.367}$$

Since we have

$$S_j[t^{-2}, a, b] = \frac{1}{\lambda_j s \sinh(\sqrt{\lambda_j}/s)}\Big|_{s=a}^{s=b}, \tag{2.368}$$

we obtain

$$-r\sinh(\sqrt{\lambda_j}/r)S_j[t^{-2}, r_0, r] + (\lambda_j^{\frac{1}{2}} + \frac{\lambda_j^{\frac{3}{2}}}{6r^2})S_j[t^{-2}, r_0, \infty] \tag{2.369}$$

$$=\frac{r\sinh(\sqrt{\lambda_j}/r)}{\lambda_j r_0 \sinh(\sqrt{\lambda_j}/r_0)} - \frac{1}{\lambda_j} + (\lambda_j^{\frac{1}{2}} + \frac{\lambda_j^{\frac{3}{2}}}{6r^2})\left(\lambda_j^{-\frac{3}{2}} - \frac{1}{\lambda_j r_0 \sinh(\sqrt{\lambda_j}/r_0)}\right) \tag{2.370}$$

$$=\frac{1}{6r^2} - \frac{\lambda_j^{\frac{1}{2}} + \frac{\lambda_j^{\frac{3}{2}}}{6r^2} - r\sinh(\sqrt{\lambda_j}/r)}{\lambda_j r_0 \sinh(\sqrt{\lambda_j}/r_0)}. \tag{2.371}$$

Therefore, we have

$$(2.366)$$

$$=\sum_{j=1}^{\infty}\chi_j\Bigg[-r\sinh(\sqrt{\lambda_j}/r)S_j[h_j(t) - \frac{15A_j}{24t^2}, r_0, r] \tag{2.372}$$

$$+ (\lambda_j^{\frac{1}{2}} + \frac{\lambda_j^{\frac{3}{2}}}{6r^2})S_j[h_j(t) - \frac{15A_j}{24t^2}, r_0, \infty] - \frac{\lambda_j^{\frac{1}{2}} + \frac{\lambda_j^{\frac{3}{2}}}{6r^2} - r\sinh(\sqrt{\lambda_j}/r)}{\lambda_j r_0 \sinh(\sqrt{\lambda_j}/r_0)} \cdot \frac{15}{4}A_j\Bigg]. \tag{2.373}$$

$$=\sum_{j=1}^{\infty}\chi_j\Bigg[\big(\lambda_j^{\frac{1}{2}} + \frac{\lambda_j^{\frac{3}{2}}}{6r^2} - r\sinh(\sqrt{\lambda_j}/r)\big)S_j[h_j(t) - \frac{15A_j}{24t^2}, r_0, r] \tag{2.374}$$

$$+ (\lambda_j^{\frac{1}{2}} + \frac{\lambda_j^{\frac{3}{2}}}{6r^2})S_j[h_j(t) - \frac{15A_j}{24t^2}, r, \infty] + \frac{r\sinh(\sqrt{\lambda_j}/r) - \lambda_j^{\frac{1}{2}} - \frac{\lambda_j^{\frac{3}{2}}}{6r^2}}{\lambda_j r_0 \sinh(\sqrt{\lambda_j}/r_0)} \cdot \frac{15}{4}A_j\Bigg]. \tag{2.375}$$

We denote with $\check{E}_j^{(1)}$, $\check{E}_j^{(2)}$ and $\check{E}_j^{(3)}$ be the three terms in line (2.374) and (2.375). According to Corollary 31, we have the estimate $h - \frac{15A}{4r^2} = O(|v|^{-\frac{1}{2}}r^{-\frac{5}{2}})$. Thus, we obtain $|h - \frac{15A}{4r^2}|_{L^2(S^2)} = O(r^{-3+\varepsilon})$. Therefore, similar to Proposition 27, we have

$$\sum_{j=1}^{\infty}(\check{E}_j^{(1)})^2 \tag{2.376}$$

$$\leq\sum_{j=1}^{\infty}\left(\int_{r_0}^{r}\frac{s^3}{r^4}\int_s^{\infty}|h_j(t)-\frac{15A_j}{24t^2}|\cdot\frac{s}{t^2}dtds\right)^2 \tag{2.377}$$

$$\leq\sum_{j=1}^{\infty}\int_{r_0}^{r}\frac{s}{r^8}ds\cdot\int_{r_0}^{r}s^5\left(\int_s^{\infty}|h_j(t)-\frac{15A_j}{24t^2}|\cdot\frac{s}{t^2}dt\right)^2ds \tag{2.378}$$

$$\leq r^{-8}(r^2-r_0^2)\int_{r_0}^{r}s^7\left(\int_s^{\infty}t^{-2}dt\cdot\int_s^{\infty}t^{-2}\sum_{j=1}^{\infty}|h_j(t)-\frac{15A_j}{24t^2}|^2dt\right)ds \tag{2.379}$$

$$=O(r^{-6+2\varepsilon}). \tag{2.380}$$

Since $(\lambda_j^{\frac{1}{2}} + \frac{\lambda_j^{\frac{3}{2}}}{6r^2})(\sinh(\lambda_j^{\frac{1}{2}}s^{-1}))^{-1} \leq s + \frac{s^3}{r^2}$, we obtain similar to the estimate of $E_j^{(3)}$ in Proposition 27, we have

$$\sum_{j=1}^{\infty}(\check{E}_j^{(2)})^2 = O(r^{-6+2\varepsilon}). \tag{2.381}$$

Since

$$0 < \frac{r\sinh(\sqrt{\lambda_j}/r) - \lambda_j^{\frac{1}{2}} - \frac{\lambda_j^{\frac{3}{2}}}{6r^2}}{\lambda_j r_0\sinh(\sqrt{\lambda_j}/r_0)} \leq \frac{r_0^4}{\lambda_j r^4}, \tag{2.382}$$

we obtain

$$\sum_{j=1}^{\infty}(\check{E}_j^{(3)})^2 = O(r^{-8}). \tag{2.383}$$

Therefore, we have $|\hat{\psi}|_{L^2(S^2)} = O(r^{-3+\varepsilon})$. Similar to Proposition 28 and 29, we have

$$\sum_{j=1}^{\infty}\lambda_j^{-1}\langle\partial_r\hat{\psi},\chi_j\rangle^2 = O(r^{-8+2\varepsilon}), \text{ and } \sum_{j=1}^{\infty}\lambda_j^{-2}\langle\partial_r^2\hat{\psi},\chi_j\rangle^2 = O(r^{-10+2\varepsilon}), \tag{2.384}$$

which finishes the proof. □

2.4.5 Improved regularity outside the north pole

Observe that $v \to 0$ at the north pole $\theta = \pi$ which leads to regularity issues for A and B. However, we are still able to show smoothness away from $\theta = \pi$ by using Neumann eigenfunctions on $S^2_\delta = \{(\theta, \varphi) \in S^2 | 0 \leq \theta \leq \pi - \delta\}$ in our expansion. These Neumann eigenfunctions are defined as follows:

$$\Delta_{S^2} \tilde{\chi}_i + \tilde{\lambda}_i \tilde{\chi}_i = 0, \text{ on } S^2_\delta, \tag{2.385}$$

$$\partial_\nu \tilde{\chi}_i = 0, \text{ on } \partial S^2_\delta \tag{2.386}$$

where ν is the outer normal to the sphere S^1_δ. We have the following characterization via the Raleigh quotient:

$$\tilde{\lambda}_0 = 0, \ \tilde{\lambda}_1 = \inf_{\chi \in H^1(S^2_\delta), \langle \chi, 1 \rangle = 0} \frac{\langle \nabla \chi, \nabla \chi \rangle}{\langle \chi, \chi \rangle}. \tag{2.387}$$

Proposition 33. *The eigenfunction expansion* $P_N \psi = \sum_{i=0}^N \langle \psi, \tilde{\chi}_i \rangle \tilde{\chi}_i$ *converges to* ψ *in* L^2 *on* S^2_δ *for* $N \to \infty$.

Proof. From the Raleigh quotient, we know that

$$\tilde{\lambda}_{N+1} \|\psi - P_N \psi\|^2_{L^2(S^2_\delta)} \tag{2.388}$$

$$\leq \|\nabla^{S^2}(\psi - P_N \psi)\|^2_{L^2(S^2_\delta)} \tag{2.389}$$

$$= - \langle \psi - P_N \psi, \Delta_{S^2}(\psi - P_N \psi) \rangle_{S^2_\delta} + \int_{\partial S^2_\delta} (\psi - P_N \psi) \partial_\nu (\psi - P_N \psi) dA \tag{2.390}$$

Observe that $\Delta_{S^2} P_N = P_N \Delta_{S^2}$. Thus,

$$\tilde{\lambda}_{N+1} \|\psi - P_N\psi\|^2_{L^2(S^2_\delta)} \tag{2.391}$$

$$\leq - \langle \psi - P_N\psi, \Delta_{S^2}(\psi - P_N\psi)\rangle + \int_{\partial S^2_\delta} (\psi - P_N\psi)\partial_\nu(\psi - P_N\psi) dA \tag{2.392}$$

$$= \langle \nabla^{S^2}(\psi - P_N\psi), \nabla^{S^2}\psi\rangle - \int_{\partial S^2_\delta} (\psi - P_N\psi)\partial_\nu\psi dA + \int_{\partial S^2_\delta} (\psi - P_N\psi)\partial_\nu\psi dA \tag{2.393}$$

$$\leq \frac{1}{2}\|\nabla^{S^2}(\psi - P_N\psi)\|^2_{L^2(S^2_\delta)} + \frac{1}{2}\|\nabla^{S^2}\psi\|^2_{L^2(S^2_\delta)}. \tag{2.394}$$

Therefore, we have

$$\|\psi - P_N\psi\|^2_{L^2(S^2_\delta)} \leq \frac{1}{\tilde{\lambda}_{N+1}} \|\nabla^{S^2}\psi\|^2_{L^2(S^2_\delta)} \to 0 \tag{2.395}$$

which finishes the proof. □

We also have higher order convergence:

Proposition 34. *Let $0 < \delta < \delta'$, then $P_N\psi \to \psi$ in $H^k(S^2_{\delta'})$ for any $k \geq 0$.*

Proof. We estimate

$$\tilde{\lambda}_{N+1} \|\Delta_{S^2}\psi - P_N\Delta_{S^2}\psi\|^2_{L^2(S^2_\delta)} \tag{2.396}$$

$$\leq \|\nabla^{S^2}(\Delta_{S^2}\psi - P_N\Delta_{S^2}\psi)\|^2_{L^2(S^2_\delta)} \tag{2.397}$$

$$= - \langle \Delta_{S^2}\psi - P_N\Delta_{S^2}\psi, \Delta_{S^2}(\Delta_{S^2}\psi - P_N\Delta_{S^2}\psi)\rangle_{S^2_\delta} \tag{2.398}$$

$$+ \int_{\partial S^2_\delta} (\Delta_{S^2}\psi - P_N\Delta_{S^2}\psi)\partial_\nu(\Delta_{S^2}\psi - P_N\Delta_{S^2}\psi) dA \tag{2.399}$$

$$= \langle \nabla^{S^2}(\Delta_{S^2}\psi - P_N\Delta_{S^2}\psi), \nabla^{S^2}\Delta_{S^2}\psi\rangle_{S^2_\delta} \tag{2.400}$$

$$\leq \|\nabla^{S^2}(\Delta_{S^2}\psi - P_N\Delta_{S^2}\psi)\|_{L^2(S^2_\delta)} \cdot \|\nabla^{S^2}\Delta_{S^2}\psi\|_{L^2(S^2_\delta)}. \tag{2.401}$$

Thus, we have

$$\|\Delta_{S^2}\psi - P_N\Delta_{S^2}\psi\|_{L^2(S^2_\delta)} \leq \frac{1}{\tilde{\lambda}_{N+1}}\|\nabla^{S^2}\Delta_{S^2}\psi\|_{L^2(S^2_\delta)} \to 0. \tag{2.402}$$

Furthermore, this implies that $P_N\psi \to \psi$ in $H^2(S^2_{\delta'})$, for $0 < \delta < \delta'$. To see this, we use standard elliptic estimates to obtain for $N \to \infty$

$$\|(\nabla^{S^2})^2(\psi - P_N\psi)\|_{L^2(S^2_{\delta'})} \leq C(\|\Delta_{S^2}(\psi - P_N\psi)\|_{L^2(S^2_\delta)} + \|\psi - P_N\psi\|_{L^2(S^2_\delta)}) \to 0. \tag{2.403}$$

Similarly, we can get

$$\|(\Delta_{S^2})^k\psi - P_N(\Delta_{S^2})^k\psi\|_{L^2(S^2_\delta)} \to 0 \tag{2.404}$$

which implies higher order convergence. □

Thus, all our previous estimates are still valid when we replace χ_j by $\tilde{\chi}_j$. We can now bootstrap the decay rate of h by applying Proposition 23, 27, 28, and 29.

Proposition 35. *Away from $\theta = \pi$, for any $d \geq 0$,*

$$\sum_{j=1}^{\infty} \tilde{\lambda}_j^d \tilde{h}_j^2(r) = O(r^{-4}) \tag{2.405}$$

where $\tilde{h}_j = \langle \tilde{\chi}_j, h \rangle$. Moreover, away from $\theta = \pi$, A and $\tilde{\psi}$ are smooth.

Proof. Corollary 31 obtain $|h| = O(r^{-2})$. Thus, estimate 2.405 is clearly satisfied for $d = 0$. We show how we can bootstrap from here.

According to the C^k estimate of h, cf. Lemma 22 , we have

$$\sum_{j=1}^{\infty} \lambda_j^{\mathfrak{a}} \tilde{h}_j^2(r) = O(r^{-2+2\mathfrak{a}}), \quad \text{for any } \ \mathfrak{a} \geq 0. \tag{2.406}$$

We want to interpolate between d and $\mathfrak{a}$. For $t \in [0,1]$ we have

$$\sum_{j=1}^{\infty} \lambda_j^{td+(1-t)\mathfrak{a}} \tilde{h}_j^2(r) \leq \left(\sum_{j=1}^{\infty} \lambda_j^{d} \tilde{h}_j^2(r) \right)^t \left(\sum_{j=1}^{\infty} \lambda_j^{\mathfrak{a}} \tilde{h}_j^2(r) \right)^{1-t} = O(r^{-4+2(1-t)(1+\mathfrak{a})}). \tag{2.407}$$

Suppose, $-4+2(1-t)(1+\mathfrak{a}) = 2-2\varepsilon < 2$ for some $\varepsilon > 0$. Then Equation (2.407) becomes

$$\sum_{j=1}^{\infty} \lambda_j^{td+2+t-\varepsilon} \tilde{h}_j^2(r) = O(r^{2-2\varepsilon}). \tag{2.408}$$

We pick $\mathfrak{a}$ sufficiently large, $\varepsilon > 0$ sufficiently small and t is close to 1 with $td+t-\varepsilon > d$. Next, we interpolate between d and $td+3-\varepsilon$ to obtain

$$\sum_{j=1}^{\infty} \tilde{\lambda}_j^{td+1+t-\varepsilon} \tilde{h}_j^2(r) = O(r^{2-2\varepsilon-\frac{6-2\varepsilon}{2+t-\varepsilon-d(1-t)}}) = O(r^{-2\varepsilon}), \tag{2.409}$$

$$\sum_{j=1}^{\infty} \tilde{\lambda}_j^{td+t-\varepsilon} \tilde{h}_j^2(r) = O(r^{2-2\varepsilon-\frac{2(6-2\varepsilon)}{2+t-\varepsilon-d(1-t)}}) = O(r^{-2-2\varepsilon}). \tag{2.410}$$

Therefore, Equation (2.408), (2.409) and (2.410) satisfy the assumptions in Proposition 29. Thus,

$$\sum_{j=1}^{\infty} \tilde{\lambda}_j^{td+t-\varepsilon} (\partial_r^2 \tilde{\gamma}_j)^2 = O(r^{-6}), \tag{2.411}$$

where $\tilde{\gamma}_j = \langle \gamma, \tilde{\chi}_j \rangle$.

According to Proposition 23, 27 and 28, we have

$$A \in H^{td+1+t-\varepsilon}(S_\delta^2) \tag{2.412}$$

and

$$\sum_{j=1}^{\infty} \tilde{\lambda}_j^{td+1+t-\varepsilon} \tilde{\gamma}_j^2 = O(r^{-2-2\varepsilon}), \quad \sum_{j=1}^{\infty} \tilde{\lambda}_j^{td+1+t-\varepsilon} (\partial_r \tilde{\gamma}_j)^2 = O(r^{-4-2\varepsilon}). \tag{2.413}$$

Equipped with the estimates above, we conclude:

$$\sum_{j=1}^{\infty} \tilde{\lambda}_j^{td+t-\varepsilon} \tilde{h}_j^2(r) = O(r^{-4}). \tag{2.414}$$

Thus, if estimate (2.405) holds for d , it also holds for $td + t - \varepsilon$ where t is close to 1. Hence, we obtain estimate (2.405) for arbitrary d by iterating this process.

To obtain smoothness we simply combine the above estimate with Lemma 23. □

Lemma 36. *Away from $\theta = \pi$, we have $\hat{\psi} = O_{2,\alpha}(r^{-3})$.*

Proof. In Proposition 35, we have shown improved estimates for h away from the north pole $\theta = \pi$. Plugging these estimates for h into Proposition 27, 28, and 29, we obtain, for any $d \geq 0$,

$$\sum_{j=1}^{\infty} \tilde{\lambda}_j^d \tilde{\gamma}_j^2 = O(r^{-4}), \ \sum_{j=1}^{\infty} \tilde{\lambda}_j^d (\partial_r \tilde{\gamma}_j)^2 = O(r^{-6}), \ \sum_{j=1}^{\infty} \tilde{\lambda}_j^d (\partial_r^2 \tilde{\gamma}_j)^2 = O(r^{-8}). \tag{2.415}$$

Combining this with our estimates for A, $\tilde{\psi}_1$ and $\tilde{\psi}_2$, cf. Proposition 24, 25, and 35, we obtain, for any $d \geq 0$,

$$\sum_{j=1}^{\infty} \tilde{\lambda}_j^d \langle h - \frac{15}{4} t^{-2} \psi, \tilde{\chi}_j \rangle^2 = O(r^{-6}). \tag{2.416}$$

Therefore, in Proposition 32, we can absorb any power of $\tilde{\lambda}_j$ into $h - \frac{15}{4} t^{-2} A$ in the computations from line (2.373)-(2.375). Then, we obtain

$$\sum_{j=1}^{\infty} \tilde{\lambda}_j^d \langle \hat{\psi}, \tilde{\chi}_j \rangle^2 = O(r^{-6}).$$

We get $O(r^{-6})$, since we are away from $\theta = \pi$. Therefore, we may use Sobolev's embedding theorem to deduce $|\hat{\psi}|_{L^\infty} = O(r^{-3})$ away from $\theta = \pi$. Since $\hat{\psi}$ satisfies

$$\Delta \hat{\psi} - 3 \frac{\nabla t}{t} \cdot \nabla \hat{\psi} = O(r^{-3}), \tag{2.417}$$

we have $\hat{\psi} = O_{2,\alpha}(r^{-3})$ away from $\theta = \pi$. □

Combining now the above Lemma and Proposition 35 with Proposition 32 finishes the proof of the first part of Theorem 12.

2.4.6 Expansion in the annular region

In this section, we prove the second part of Theorem 12. Recall that the general asymptotics are given by

$$|g - b|_b = O_{2,\alpha}(r^{-\tau}), \ |k - b|_b = O_{1,\alpha}(r^{-\tau}). \tag{2.418}$$

Thus, we have $\Delta_g v + K|\nabla v| = O_{1,\alpha}(|v| r^{-\tau})$. According to the global barrier estimate from the previous section, $w = O(|v| r^{-1})$ and w satisfies

$$\Delta_g w + K \frac{\nabla(w + 2v)}{|\nabla(w + v)| + |\nabla v|} \cdot \nabla w = O_{1,\alpha}(|v| r^{-\tau}). \tag{2.419}$$

Recall that

$$\mathfrak{h} = r^2 \partial^2 \phi + r^{-2} \Delta_{S^2} \phi \tag{2.420}$$

where

$$\begin{aligned} \mathfrak{h} =& r^2 \partial^2 \phi + r^{-2} \Delta_{S^2} \phi & (2.421) \\ =& -\partial_r^2 \phi - \frac{2}{r} \partial_r \phi - \left[\frac{\bar{\nabla} t}{t} - \frac{\bar{\nabla} |v|}{|v|} \right] \cdot \bar{\nabla} \phi + |v|^{-1} t^{-1} \phi + O_1(r^{1-\tau}). & (2.422) \end{aligned}$$

Lemma 37. *We have*

$$\int_{S^2} \mathfrak{h}^2 d\sigma = O(r^{2-2\tau}). \tag{2.423}$$

Proof. We compute

$$\int_{S^2} (|v|^{-1} t^{-1} \phi)^2 d\sigma \leq C \int_{S^2} |v|^{-2} t^{-2} d\sigma = O(r^{-2}), \tag{2.424}$$

and

$$\begin{aligned} & \int_{S^2} \left(\left[\frac{\bar{\nabla} t}{t} - \frac{\bar{\nabla} |v|}{|v|} \right] \cdot \bar{\nabla} \phi \right)^2 d\sigma & (2.425) \\ \leq & C \int_{S^2} \left| \frac{\bar{\nabla} t}{t} - \frac{\bar{\nabla} |v|}{|v|} \right|^2 d\sigma & (2.426) \\ = & C \int_{S^2} \frac{2}{t|v|} - \frac{1}{t^2} d\sigma & (2.427) \\ = & O(r^{-2} \ln r) & (2.428) \end{aligned}$$

where we used that $\phi = O_3(1)$. Since $\tau > \frac{3}{2}$, we have $2 - 2\tau > -2$. Thus, the lemma is proven. □

Proof of the second part of Theorem 12. Using again separation of variables we obtain the decomposition

$$\phi = \hat{A} + \hat{\phi}. \tag{2.429}$$

Therefore, $\hat{A} \in H^{\tau-\varepsilon-1}$, and for $0 \leq d < \tau - 1$ we have

$$\sum_{j\neq 0} \lambda_j^d \langle \hat{\phi}, \chi_j \rangle^2 d\sigma = O(r^{2-2\tau+2d}) \tag{2.430}$$

where χ_j are the eigenfunctions of Δ on S^2. Since $\int_{S^2} \mathfrak{h}^2 d\sigma = O(r^{2-2\tau})$, and $\phi = O_3(|v|r^{-1})$, we have $\int_{S^2} |\nabla^{S^2}\mathfrak{h}|^2 d\sigma = O(r^{4-2\tau})$. Moreover, using Proposition 28, 24 and 25 we have

$$\int_{S^2} (\partial_r \hat{\phi})^2 d\sigma = O(r^{-2\tau}). \tag{2.431}$$

This concludes the proof of of Theorem 12. □

We end this section by proving two more estimates for the expansion in the annular region will be used in the mass computation in the next section.

Proposition 38. *For some small $\varepsilon > 0$, we have $\int_{S^2} |\partial_r w| d\sigma = O(r^{\tau-3-\varepsilon})$.*

Proof. Since $|\bar{\nabla}|v|t^{-1}|^2 = 2|v|t^{-3} - |v|^2 t^{-4} = O(|v|r^{-3})$, we have $\partial_r(|v|r^{-1}) = O(r^{-2})$.

Next, we estimate

$$\int_{S^2} |\partial_r w| d\sigma = \int_{S^2} |\partial_r(|v|r^{-1}) + |v|r^{-1}\partial_r \phi| d\sigma \tag{2.432}$$

$$\leq C \int_{S^2} |\partial_r \phi| d\sigma + O(r^{-2}) \tag{2.433}$$

$$= C \int_{S^2} |\partial_r \hat{\phi}| d\sigma + O(r^{-2}) \tag{2.434}$$

$$\leq C \left(\int_{S^2} |\partial_r \hat{\phi}|^2 d\sigma \right)^{\frac{1}{2}} + O(r^{-2}) \tag{2.435}$$

$$= O(r^{-\tau}). \tag{2.436}$$

We pick a small $\varepsilon > 0$ such that $\tau - 3 - \varepsilon > -\tau$, and the result follows. □

Proposition 39. *For any* $0 < \varepsilon < \tau - 1$*, we have* $\sum_{j=1}^{\infty} \lambda_j^{\tau-\varepsilon-1} \langle \phi, \chi_j \rangle^2 = O(1)$*, i.e.,* $\|\phi\|_{H^{\tau-\varepsilon-1}(S^2)} = O(1)$*.*

Proof. Since $\phi = \hat{A} + \hat{\phi}$ and $\hat{A} \in H^{\tau-\varepsilon-1}(S^2)$, the result follows from estimate (2.430).

□

Proposition 40. *Let* $\phi = \hat{A} + \hat{\phi}$ *Then, we have*

$$\int_{S^2} |\nabla u| - |\nabla v| d\sigma = O(r^{\tau-2-\varepsilon}). \tag{2.437}$$

Proof. Since $w = O_1(|v|r^{-1})$, then

$$|\nabla u| = |\nabla(w + v)| \tag{2.438}$$

$$= \sqrt{|\nabla v|^2 + 2\nabla v \cdot \nabla w + |\nabla w|^2} \tag{2.439}$$

$$= |\nabla v| \left(1 + \frac{\nabla v \cdot \nabla w}{|\nabla v|^2} + O(r^{-2}) \right). \tag{2.440}$$

Therefore, using $|\nabla^{S^2}|v||_{C^0(S^2)} = O(|v|^{\frac{1}{2}}r^{\frac{1}{2}})$ and $|\nabla w| = O(|v|r^{-1})$, we have

$$\begin{aligned}
\int_{S^2} |\nabla u| - |\nabla v| d\sigma \leq & \int_{S^2} \frac{\nabla v \cdot \nabla w}{|\nabla v|} + O(|v|r^{-2}) d\sigma && (2.441)\\
= & \int_{S^2} \frac{\nabla v}{|\nabla v|} \cdot \partial_r w \nabla r + r^{-2} \frac{\nabla^{S^2}|v|}{|\nabla v|} \cdot \nabla^{S^2} w + O(|v|r^{-2}) d\sigma && (2.442)\\
\leq & C \int_{S^2} r|\partial_r w| + |v|^{-\frac{1}{2}} r^{-\frac{1}{2}} |\nabla w| d\sigma + O(r^{-1}) && (2.443)\\
\leq & C \int_{S^2} |v|^{\frac{1}{2}} r^{-\frac{3}{2}} d\sigma + O(r^{\tau-2-\varepsilon} + r^{-1}) && (2.444)\\
= & O(r^{\tau-2-\varepsilon}). && (2.445)
\end{aligned}$$

This proves the proposition. □

2.5 Obtaining energy and momentum in the interpolation region

The goal of this section is to show that we indeed obtain the energy and momentum from the annulus integral in our mass formula. We begin with studying the definition of energy and momentum from the previous section more closely.

2.5.1 Definition of spacetime energy and momentum revisited

In this section we show that our definition of mass can be recovered from Chruściel-Jezierski-Leski's one.

Let V be a function satisfying $\bar{\nabla}_{ij}V = Vb_{ij}$ where b is the hyperbolic metric and $\bar{\nabla}$ the connection with respect to b. The general definition of mass from Chruściel, Jezierski and Leski states in our setting

$$H(V) = \lim_{r\to\infty}\int_{S_r}[\mathbb{U}^i(V) + \mathbb{V}^i(V)]dS_i, \tag{2.446}$$

where $\mathbb{U}^i(V)$, $\mathbb{V}^i(V)$ are defined by

$$\mathbb{U}^i(V) = 2\sqrt{\det g}(Vg^{i[k}g^{j]l}\bar{\nabla}_j g_{kl} + \bar{\nabla}^{[i}Vg^{j]k}h_{jk}), \tag{2.447}$$

$$\mathbb{V}^l(V) = 2\sqrt{\det g}(P^l_k - \bar{P}^l_k)\bar{\nabla}^k V. \tag{2.448}$$

Here $h = g - b$, $P^{ij} = g^{ij}\mathrm{tr}_g k - k^{ij}$ and $g^{i[k}g^{j]l} = \frac{1}{2}(g^{ik}g^{jl} - g^{ij}g^{kl})$. Note that[1] $J_l = -D_i P^i_l$ and $P^{ij} - \bar{P}^{ij} = O_{l-1,\alpha}(r^{-\tau})$, where $\bar{P}^{ij} = b^{ij}\mathrm{tr}_b b - b^{ij} = 2b^{ij}$. Next, we show that this definition from [CJL04] recovers our definition from the previous

[1] Our J has a different sign as [CJL04].

section.

In [BM11], Proposition 4, a formula for the scalar curvature under small perturbations of the metric has been computed. It states

$$\left| R_g - \bar{R} + \langle \overline{Ric}, h \rangle + \bar{\nabla}_i (g^{ik} g^{jl} (\bar{\nabla}_k h_{jl} - \bar{\nabla}_l h_{kj})) \right| \leq C |\bar{\nabla} h|^2 + C|h|^2, \tag{2.449}$$

where $\bar{R} = -6$ is the scalar curvature of $(\mathbb{H}^3, b)$. We obtain

$$\begin{aligned} \partial_i \mathbb{U}^i(V) =& 2\sqrt{\det g}(\bar{\nabla}_i V g^{i[k} g^{j]l} \bar{\nabla}_j g_{kl} + V \bar{\nabla}_i g^{i[k} g^{j]l} \bar{\nabla}_j g_{kl}) & (2.450) \\ & + \sqrt{\det g} V \mathrm{tr}_g h + 2\sqrt{\det g} \bar{\nabla}^i V g^{jk} \bar{\nabla}_{[i} h_{j]k} + \sqrt{\det g} V O_{l-1,\alpha}(r^{-2\tau}) & (2.451) \\ =& \sqrt{\det g} V (R_g - \bar{R}) + \sqrt{\det g} V O_{l-1,\alpha}(r^{-2\tau}), & (2.452) \end{aligned}$$

Since $\bar{\nabla}_l (\bar{P})^l_k = 0$, $\bar{\nabla}_l (P^l_k - \bar{P}^l_k) = \nabla_l P^l_k + O_{l-1,\beta}(r^{-2\tau}) = -J + O_{l-1,\beta}(r^{-2\tau})$, and $2\mathrm{tr}_g k - 3 = \frac{1}{2}[(\mathrm{tr}_g k)^2 - |k|^2] + O_{l-1,\beta}(r^{-2\tau})$, we have

$$\begin{aligned} & \partial_l \mathbb{V}^l(V) & (2.453) \\ =& 2\sqrt{\det g}[\bar{\nabla}_l (P^l_k - \bar{P}^l_k)] \bar{\nabla}^k V + 2\sqrt{\det g}(2\mathrm{tr}_g k - 6) V + \sqrt{\det g} V O_{l-1,\alpha}(r^{-2\tau}) & (2.454) \\ =& -2\sqrt{\det g} \langle J, \nabla V \rangle + \sqrt{\det g}[(\mathrm{tr}_g k)^2 - |k|^2 - 6] V + \sqrt{\det g} V O_{l-1,\alpha}(r^{-2\tau}). & (2.455) \end{aligned}$$

Therefore,

$$\partial_i (\mathbb{U}^i(V) + \mathbb{V}^i(V)) = 2\sqrt{\det g}\, [V\mu - \langle J, \nabla V \rangle] + \sqrt{\det g} V O_{l-1,\alpha}(r^{-2\tau}) \tag{2.456}$$

where $\mu = \frac{1}{2}(R_g - |k|^2 + (\mathrm{tr}_g k)^2)$. Thus, we can also express the mass functional by

$h = g - b$ and $p := k - g$:

$$H_\Phi(V) = \lim_{r\to\infty} \int_{S_r} \left[V\left(\mathrm{div}_b h - d(\mathrm{tr}_b h)\right) + \mathrm{tr}_b(h+2p)dV - (h+2p)(\bar{\nabla} V, \cdot)\right](\nu_\rho) d\mu_\sigma,$$

where $\nu_\rho = \sqrt{1+r^2}\partial_r$ is the outward unit normal vector on S_r. This recovers our energy and momentum definition from the previous section by setting

$$E = \frac{1}{16\pi} H_\Phi(V_0), \quad P_i = \frac{1}{16\pi} H_\Phi(V_i), \quad i = 1,2,3.$$

where $V_0 = \sqrt{r^2+1}$ and $V_i = x_i$, $i = 1,2,3$.

2.5.2 Recovering the mass in the annular interpolation region

In this section, we show that $\int_{Annulus} \mu|\nabla u| + \langle J, \nabla u\rangle$ is indeed recovering the energy and momentum.

Since (M, g, k) is an asymptotically hyperbolic manifold with decay rate τ, $\frac{3}{2} < \tau \leq 3$, we have

$$g = b + O_2(r^{-\tau}),\ k = g + O_1(r^{-\tau}). \tag{2.457}$$

We define the interpolation metric and second fundamental form $\check{g}$ and $\check{k}$ in the interpolation region $I = \{r < \rho < 2r\}$ by

$$\check{g} = \eta(\rho)g + (1-\eta(\rho))b,\ \check{k} = \eta(\rho)k + (1-\eta(\rho))b, \tag{2.458}$$

where $\eta(\rho)$ is the cutoff function satisfying $\eta(r) = 1$, $\eta(2r) = 0$ and

$$\rho\|\partial_\rho \eta\|_{C^0} + \rho^2\|\partial_\rho^2 \eta\|_{C^0} \leq C. \tag{2.459}$$

Then we have

$$\check{g} = b + O_2(r^{-\tau}),\ \check{k} = b + O_1(r^{-\tau}). \tag{2.460}$$

Therefore, the energy and momentum density μ^r and J^r satisfy in the interpolation region

$$|\mu^r| + |J^r| = O(r^{-\tau}). \tag{2.461}$$

Moreover, we obtain an improved decay estimate for $J^r - \eta J$.

Lemma 41. *$J^r - \eta J = O_1(r^{-\tau}) + O(r^{-2\tau})$.*

Proof. Let $\check{p} = \check{k} - \check{g}$. Then $\check{p} = \eta(\rho)p = O_{1,\alpha}(r^{-\tau})$ where $p = k - g$. Let e_l be a unit vector and denote $J^r_l = \langle J^r, e_l\rangle_{\check{g}}$. We compute

$$J^r_l = (\mathrm{div}_{\check{g}}\check{p} - d\mathrm{tr}_{\check{g}}(\check{p}))_l \tag{2.462}$$

$$= \check{g}^{ij}\check{\nabla}_i \check{p}_{jl} - \check{g}^{ij}\check{\nabla}_l \check{p}_{ij} \tag{2.463}$$

$$= \eta\left(\check{g}^{ij}\check{\nabla}_i p_{jl} - \check{g}^{ij}\check{\nabla}_l p_{ij}\right) + \check{g}^{ij}p_{jl}\partial_i\eta - \check{g}^{ij}\partial_l\eta \tag{2.464}$$

$$= \eta[\check{g}^{ij}\nabla_i p_{jl} - \check{g}^{ij}\nabla_l p_{ij} + O_1(r^{-\tau})] + O_1(r^{-\tau}) \tag{2.465}$$

$$= \eta[g^{ij}\nabla_i p_{jl} - g^{ij}\nabla_l p_{ij} + O(r^{-2\tau}) + O_1(r^{-\tau})] + O_1(r^{-\tau}) \tag{2.466}$$

$$= \eta J + O(r^{-2\tau}) + O_1(r^{-\tau}) \tag{2.467}$$

which finishes the proof. □

We use the estimate from the expansion section to get the following estimate.

Proposition 42. *$\int_{S^2}\langle\check{\nabla}w, J^r\rangle d\sigma = O(r^{-2-\varepsilon})$, for any small $0 < \varepsilon \leq \tau - \frac{3}{2}$.*

Proof. We first estimate the integral of the radial part of ∇w using Proposition 38. Since $J^r = O(r^{-\tau})$, we have

$$\left|\int_{S^2} \langle \partial_r w \check{\nabla} r, J^r \rangle d\sigma\right| \leq C \int_{S^2} r^{1-\tau} |\partial_r w| d\sigma = O(r^{-2-\varepsilon}), \tag{2.468}$$

According to Lemma 41, we have a decomposition $J^r - \eta J = J_1 + J_2$ where $J_1 = O_1(r^{-\tau})$, and $J_2 = O(r^{-2\tau})$. The estimate $w = O_3(|v|r^{-1})$ implies $\nabla^{S^2} w = O(r)$. Since $J \in C^{2,\alpha}_{3+\varepsilon}$, we have

$$\int_{S^2} \langle \nabla^{S^2} w, \eta J + J_2 \rangle d\sigma = O(r^{-2-\varepsilon} + r^{-2\tau+1}). \tag{2.469}$$

We use Cauchy-Schwarz inequality for Sobolev spaces with negative exponents to estimate the leftover term. Since $|\nabla^{S^2} |v| r^{-1}| = O(|v|^{\frac{1}{2}} r^{-\frac{1}{2}}) = O(1)$, $\phi = O_3(1)$, we obtain

$$\left|\int_{S^2} \langle \nabla^{S^2} w, J_1 \rangle d\sigma\right| \tag{2.470}$$

$$= \left|\int_{S^2} \langle \phi \nabla^{S^2} (|v| r^{-1}) + |v| r^{-1} \nabla^{S^2} \phi, J_1 \rangle d\sigma\right| \tag{2.471}$$

$$\leq O(r^{-\tau}) + C \left|\int_{S^2} \langle \nabla^{S^2} \phi, J_1 \rangle d\sigma\right| \tag{2.472}$$

$$\leq C \|\nabla^{S^2} \phi\|_{H^{-\frac{1}{2}}(S^2)} \|J_1\|_{H^{\frac{1}{2}}(S^2)} + O(r^{-\tau}) \tag{2.473}$$

$$= O(r^{-\tau+\frac{1}{2}}). \tag{2.474}$$

Here the last line is based on Proposition 39 and $\|J_1\|_{H^{\frac{1}{2}}(S^2)} = O(r^{-\tau+\frac{1}{2}})$ which follows from the estimate

$$\sum_{j\neq 0} \lambda^{\frac{1}{2}} \langle J_1, \chi_j \rangle^2 \leq \left(\sum_{j\neq 0} \lambda_j \langle J_1, \chi_j \rangle^2\right)^{\frac{1}{2}} \cdot \left(\sum_{j\neq 0} \langle J_1, \chi_j \rangle^2\right)^{\frac{1}{2}} = O(r^{-2\tau+1}). \tag{2.475}$$

We choose $\varepsilon \leq \tau - \frac{3}{2}$. Then the decay rate estimate in line (2.474) implies the decay rate of the integral of the tangential part is $O(r^{-\tau-\frac{1}{2}}) = O(r^{-2-\varepsilon})$. □

Theorem 43. *For each $r \gg 1$, let $\check{g}$ and $\check{k}$ be the interpolated metric and second fundamental form in the annulus $M_{2r} \setminus M_r$. Moreover, for each r, let u be the spacetime harmonic function on $(M, \check{g}, \check{k})$ which is asymptotic to $-x - t$. Denote $I = M_{2r} \setminus M_r$. Then*

$$\int_I \mu^r |\check{\nabla} u| + \langle J^r, \check{\nabla} u \rangle \to -8\pi(E + \langle P, \partial_x \rangle) \tag{2.476}$$

for $r \to \infty$.

Proof. Recall that Proposition 40 implies

$$\int_{S^2} |\check{\nabla} u| - |\check{\nabla} v| d\sigma = O(r^{\tau-2-\varepsilon}). \tag{2.477}$$

Thus, we can compute

$$\int_I 2(|\check{\nabla} u|\mu^r + \langle J^r, \check{\nabla} u \rangle) dV_{\check{g}} \tag{2.478}$$

$$= \int_I 2(|\check{\nabla} v|\mu^r + \langle J^r, \check{\nabla} v \rangle) dV_{\check{g}} + \int_{C_1 r}^{C_2 r} O(r^{-1-\varepsilon}) dR \tag{2.479}$$

$$= \int_{\partial_+ I} \partial_i (\mathbb{U}^i(|v|) + \mathbb{V}^i(|v|)) dS_i - \int_{\partial_- I} \partial_i (\mathbb{U}^i(|v|) + \mathbb{V}^i(|v|)) S_i + O(r^{-\varepsilon}), \tag{2.480}$$

where the last inequality follows from line (2.456).

Since the metric in $\{R \geq 2r\}$ is exactly hyperbolic, then $\mathbb{U}^i(|v|) = \mathbb{V}^i(|v|) = 0$ on $\partial_+ I = \{R = 2r\}$, hence,

$$\int_{\partial_+ I} \partial_i (\mathbb{U}^i(|v|) + \mathbb{V}^i(|v|)) dS_i = 0. \tag{2.481}$$

When $r \to \infty$, on $\partial\_I = \{R = r\}$, we have:

$$\int_{\partial\_I} \partial_i(\mathbb{U}^i(|v|) + \mathbb{V}^i(|v|))dS_i \to H(|v|). \tag{2.482}$$

Therefore, when $r \to \infty$,

$$\int_I 2(|\nabla u|\mu^r + \langle J^r, \nabla u\rangle)dV_g \to -H(|v|) \tag{2.483}$$

as desired. □

For each interpolation $M_{2r}\backslash M_r$, we have the spacetime harmonic function u^r. The C^0 of u^r is uniform in r, then combining with the standard elliptic estimate, we have a uniform $C^{2,\alpha}$ estimate of u^r. Therefore, we take a subsequent of u^r with the limit u. Subsequently, we have

$$\lim_{r\to\infty} \int_{M_{2r}\backslash M_r} 2(|\nabla u^r|\mu^r + \langle J^r, \nabla u^r\rangle)dV_g \to H(v). \tag{2.484}$$

Hence, we focus on the interpolated manifold, then taking the limit, we obtain the main theorem.

2.6 Computations near infnity

In this section, we show that the boundary integral

$$\int_{\partial M_r} \partial_{\mathbf{n}}(|\nabla u| + u) + \int_{M_r} |\nabla u| R_\Sigma \to 0, \text{ when } r \to \infty,$$

when M is hyperbolic near infinity. For this purpose, we have to choose our exhaustion M_r of M very carefully. In fact, we define a two-paramenter family of surfaces

$$\partial\Omega = \{x_1^2 + x_2^2 + (\frac{2}{u} + 2R + \varepsilon)^2 = 4R^2\} \tag{2.485}$$

where we will let first let R go to ∞ and then ε go to 0. This particular choice of surfaces greatly simplifies the following computations of the boundary integral.

In order to compute the boundary term $\int_{\partial M_r} \nabla_{\mathbf{n}}(|\nabla u| + u) + \int_{M_r} |\nabla u| R_\Sigma$, we first compute the Gaussian curvature integral and then compute the $(\nabla_{\mathbf{n}}|\nabla u| + \nabla_i u k_{i\mathbf{n}})$ integral. Moreover, we have to split for both of these computations the surface $\partial\Omega = \{x_1^2 + x_2^2 + (\frac{2}{u} + 2R + \varepsilon)^2 = 4R^2\}$ into several regions where we employ different types of computational tricks.

2.6.1 Gaussian curvature integral

We split the integral $\int_{M_r} |\nabla u| R_\Sigma$ into two pieces, one of which where we apply Gauss-Bonnet's theorem and estimate the geodeisc curvature and one where we do not apply Gauss-Bonnet's theorem and estimate the Gaussian curvature instead. We define $S_1 = \{\varepsilon + R^{-\delta} \leq -\frac{2}{u} \leq R^\delta\} \cap \partial\Omega$ and begin with the computation of the corresponding geodeisc curvature integral.

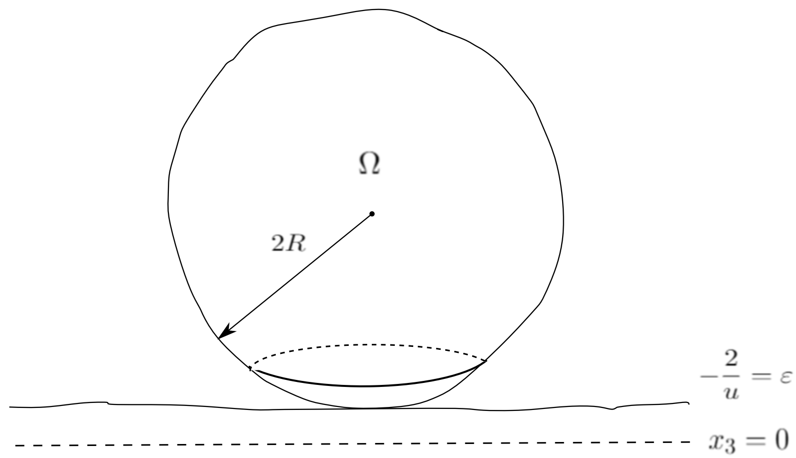

Figure 2.4: Integral Region

Estimating the geodesic curvature

According to the barrier in previous section, we have $u = v + w$, $w = O_2(|v|^{\frac{3}{2}} r^{-\frac{3}{2}})$, therefore, $c_1|v| \leq |u| \leq c_2|v|$. In the upper half space model, we have $v = -\frac{2}{x_3}$.

Let

$$\Omega = \{x_1^2 + x_2^2 + (-\frac{2}{u} - 2R - \varepsilon)^2 < 4R^2\}. \tag{2.486}$$

Suppose $\rho = \sqrt{x_1^2 + x_2^2}$. On $\partial\Omega$, we have

$$\rho = \sqrt{4R^2 - \left(\frac{2}{u} + 2R + \varepsilon\right)^2} = \sqrt{\left(-\frac{2}{u} - \varepsilon\right)\left(4R + \varepsilon + \frac{2}{u}\right)}. \tag{2.487}$$

When $\varepsilon + R^{-\delta} \leq -\frac{2}{u} \leq R^{\delta}$, we have $c_3 R^{\frac{1-\delta}{2}} \leq \rho \leq c_4 R^{\frac{1+\delta}{2}}$. Since $\sqrt{r^2+1} = \frac{\rho^2 + x_3^2 + 4}{4x_3}$, we obtain

$$\sqrt{r^2+1} \geq x_3^{-1}\rho^2 \geq c_5|u|R^{1-\delta} \geq c_6 R^{1-2\delta}. \tag{2.488}$$

Since on $\partial\Omega$, $|v| \leq 2\varepsilon^{-1}$, then, in the region $S_1 = \{\varepsilon + R^{-\delta} \leq -\frac{2}{u} \leq R^{\delta}\} \cap \partial\Omega$,

$$u = v(1 + |v|^{1/2} O_2(r^{-\frac{3}{2}})) = v(1 + O_2(r^{-l})), \ 1 < l < \frac{3}{2}, \tag{2.489}$$

where l is a fixed constant chosen to absorb $|v|^{\frac{1}{2}}$.

Set $z = -\frac{2}{u}$ and $z_i = \frac{\partial z}{\partial x_i}$. Then $z = x_3(1+O_2(r^{-l}))$, $z_1, z_2 = O_1(r^{-l})$ and $z_3 = 1+O_1(r^{-l})$. Let $X_1 = (x_1+z_1(z-2R-\varepsilon))\partial x^1+(x_2+z_2(z-2R-\varepsilon))\partial x^2+z_3(z-2R-\varepsilon)\partial x^3$. Then X_1 is an outer normal vector of $\partial\Omega$, and as $z - 2R - \varepsilon = O_1(R)$, we have:

$$\begin{aligned} X_1 =& x_1\partial x^1 + x_2\partial x^2 + (z - 2R - \varepsilon)\partial x^3 + (z - 2R - \varepsilon)x_3^{-1} O_1(r^{-l}) & (2.490)\\ =& x_1\partial x^1 + x_2\partial x^2 + (z - 2R - \varepsilon)\partial x^3 + R x_3^{-1} O_1(r^{-l}). & (2.491)\end{aligned}$$

Hence, on $\partial\Omega$

$$|X_1| = |x_1\partial x^1+x_2\partial x^2+(z-2R-\varepsilon)\partial x^3|+Rx_3^{-1}O_1(r^{-l}) = \frac{2R}{x_3}+Rx_3^{-1}O_1(r^{-l}). \tag{2.492}$$

Denoting by ν the normal vector of the level set Σ_t, we compute

$$\begin{aligned} \nu =& \frac{\nabla u}{|\nabla u|} & (2.493)\\ =& \frac{x_3^2(u_1\partial x^1 + u_2\partial x^2 + u_3\partial x^3)}{|\nabla v| + vO_1(r^{-l})} & (2.494)\\ =& \frac{2\partial x^3 + x_3^{-1}O_1(r^{-l})}{2x_3^{-1} + x_3^{-1}O_1(r^{-l})} & (2.495)\\ =& x_3\partial x^3 + O_1(r^{-l}) & (2.496)\end{aligned}$$

We set $\beta_1 = X_1 - \langle X_1, \nu\rangle\nu$ and compute

$$\beta_1 = x_1\partial x^1 + x_2\partial x^2 + (z-2R-\varepsilon)\partial x^3 + Rx_3^{-1}O_1(r^{-l}) - (z-2R-\varepsilon)\partial x^3 - Rx_3^{-1}O_1(r^{-l}) \tag{2.497}$$

$$= x_1\partial x^1 + x_2\partial x^2 + Rx_3^{-1}O_1(r^{-l}). \tag{2.498}$$

Let α_1 be a vector perpendicular to X_1 and ν, then we pick α_1 as below:

$$\alpha_1 = x_3x_2\partial x^1 - x_3x_1\partial x^2 + RO_1(r^{-l}). \tag{2.499}$$

Let $\alpha_2 = x_3^{-1}\alpha_1$, $|\alpha_2| = \rho x_3^{-1} + Rx_3^{-1}O_1(r^{-l})$.[2] Assume $\alpha_3 = x_2\partial x^1 - x_1\partial x^2$, then $\alpha_2 = \alpha_3 + Rx_3^{-1}O_1(r^{-l})$.

Here are the non-vanishing Christoffel symbols in the upper half space model:

$$\bar{\Gamma}^3_{33} = \bar{\Gamma}^2_{23} = \bar{\Gamma}^2_{32} = \bar{\Gamma}^1_{13} = \bar{\Gamma}^1_{31} = -\frac{1}{x_3},\ \bar{\Gamma}^3_{22} = \bar{\Gamma}^3_{11} = \frac{1}{x_3},$$

then

$$\nabla_{x_3\partial x^1}\alpha_3 = -x_3\partial x^2 + x_2\partial x^3,\ \nabla_{x_3\partial x^2}\alpha_3 = x_3\partial x^1 - x_1\partial x^3,\ \nabla_{x_3\partial x^3}\alpha_3 = -x_2\partial x^1 + x_1\partial x^2. \tag{2.500}$$

Hence, $|\nabla\alpha_3| = O(1+\rho x_3^{-1})$.

[2] We assume $Rx_3^{-1}O_1(r^{-l}) << \rho x_3^{-1}$

$$\nabla_{\alpha_2}\alpha_2 = \nabla_{\alpha_3}\alpha_3 + \nabla_{Rx_3^{-1}O_1(r^{-l})}\alpha_3 + \nabla_{\alpha_2} Rx_3^{-1}O_1(r^{-l}) \tag{2.501}$$

$$= -x_1\partial x^1 - x_2\partial x^2 + \frac{\rho^2}{x_3}\partial x^3 + (1+\rho x_3^{-1})Rx_3^{-1}O(r^{-l}) + \rho x_3^{-1}Rx_3^{-1}O(r^{-l}) \tag{2.502}$$

$$= -x_1\partial x^1 - x_2\partial x^2 + \frac{\rho^2}{x_3}\partial x^3 + (1+\rho x_3^{-1})Rx_3^{-1}O(r^{-l}). \tag{2.503}$$

Hence, assume $\beta = \frac{\beta_1}{|\beta_1|}$, $\alpha_0 = \frac{\alpha_2}{|\alpha_2|}$.

$$k_t = -\langle \beta, \nabla_{\alpha_0}\alpha_0\rangle \tag{2.504}$$

$$= -|\beta_1|^{-1}|\alpha_2|^{-2}\langle \beta_1, \nabla_{\alpha_2}\alpha_2\rangle \tag{2.505}$$

$$= -(\rho x_3^{-1} + Rx_3^{-1}O_1(r^{-l}))^{-1}(\rho x_3^{-1} + Rx_3^{-1}O_1(r^{-l}))^{-2}\cdot \tag{2.506}$$

$$[-x_3^{-2}\rho^2 + (\rho + \rho^2 x_3^{-1})x_3^{-1}Rx_3^{-1}O_1(r^{-l}) + \rho x_3^{-1}(\rho x_3^{-1}+1)Rx_3^{-1}O(r^{-l})] \tag{2.507}$$

$$= x_3\rho^{-1} + (\rho^{-2}x_3 + \rho^{-1})RO(r^{-l}) \tag{2.508}$$

In $\{x_1, x_2, u\}$ coordinate, we have the metric:

$$g = \frac{1}{x_3^2}(dx_1^2 + dx_2^2 + dx_3^2) \tag{2.509}$$

$$= \frac{1}{x_3^2}(dx_1^2 + dx_2^2) + \frac{1}{x_3^2}u_3^{-2}(du - u_1dx_1 - u_2dx_2)^2 \tag{2.510}$$

$$= \frac{1}{x_3^2}\left[(1+\frac{u_1^2}{u_3^2})dx_1^2 + (1+\frac{u_2^2}{u_3^2})dx_2^2 + 2\frac{u_1u_2}{u_3^2}dx_1dx_2 - \frac{2u_1}{u_3^2}dx_1dx_3 - \frac{2u_2}{u_3^2}dx_2dx_3 + \frac{1}{u_3^2}du^2\right] \tag{2.511}$$

Let $x_1 = \rho\cos\theta$, $x_2 = \rho\sin\theta$, $\gamma(\theta) = \Sigma_t \cap \partial\Omega$, then $\gamma'(\theta) = -\rho\sin\theta\partial x^1 + \rho\cos\theta\partial x^2$ is the tangent vector of $\Sigma_t \cap \partial\Omega$ in $\{x_1, x_2, u\}$ coordinate. We have the length of $\gamma'(\theta)$:

$$|\gamma'(\theta)|_{\tilde{g}} = x_3^{-1}\left[(\rho\sin\theta)^2(1+\frac{u_1^2}{u_3^2}) - 2\rho^2\sin\theta\cos\theta\frac{u_1u_2}{u_3^2} + (\rho\cos\theta)^2(1+\frac{u_2^2}{u_3^2})\right]^{1/2} \tag{2.512}$$

$$= x_3^{-1}\rho|u_3|^{-1}[u_3^2 + (u_1\sin\theta - u_2\cos\theta)^2]^{1/2} \tag{2.513}$$

$$= x_3^{-1}\rho(2x_3^{-2} + x_3^{-2}O_1(r^{-l}))^{-1}[4x_3^{-4} + x_3^{-4}O_1(r^{-l}) + x_3^{-4}O_1(r^{-2l})]^{1/2} \tag{2.514}$$

$$= x_3^{-1}\rho + x_3^{-1}\rho O_1(r^{-l}) \tag{2.515}$$

Let $S_1 = \partial\Omega \cap \{\varepsilon + R^{-\delta} \leq -\frac{2}{u} \leq R^{\delta}\}$, then $\rho = [4R^2 - (\frac{2}{u} + 2R + \varepsilon)^2]^{1/2}$, in S_1, we have

$$C_1 R^{\frac{1-\delta}{2}} \leq \rho \leq C_2 R^{\frac{1+\delta}{2}}.$$

$$\int_{S_1} k_t |du| dA = \int_{-\frac{2}{\varepsilon+R^{-\delta}}}^{-\frac{2}{R^{\delta}}} \int_0^{2\pi} |\gamma'(\theta)|_{\tilde{g}} k_t d\theta du \tag{2.516}$$

$$= \int_{-\frac{2}{\varepsilon+R^{-\delta}}}^{-\frac{2}{R^{\delta}}} \int_0^{2\pi} 1 + [(\rho^{-1} + x_3^{-1})R + 1]O(r^{-l}) d\theta du \tag{2.517}$$

In S_1, we have:

$$[(\rho^{-1} + x_3^{-1})R + 1]r^{-l} \tag{2.518}$$

$$\leq C_{3,\varepsilon} R\rho^{-2l} x_3^l \tag{2.519}$$

$$\leq C_{4,\varepsilon} R \cdot R^{-l(1-\delta)} \cdot R^{l\delta}, \tag{2.520}$$

therefore, we can select δ, $\delta > 0$ such that $1 - l(1-\delta) + l\delta < 0$.

Hence, the error term in Equation (2.517) vanishes when $R \to \infty$. To apply Gauss-Bonnet theorem, we need to control the topology of the level sets.

Lemma 44. *Let (M, g, k) be a complete, simple connected, asymptotically hyper-*

bolic manifold. Then the regular level sets $\Sigma_t := \{u = t\}$ *are connected with Euler characteristic* $\chi(\Sigma_t) \leq 1$.

Proof. If Σ_t is not connected, then Σ_t bounds a compact region. We have a contradiction, according to the maximal principle. □

Applying Sard's theorem, then the measure of the singular level set is 0. Therefore, we have

$$\liminf_{r\to\infty} \int_{\Omega_r} R_\Sigma |\nabla u| dA \leq 0. \tag{2.521}$$

Estimating the Gaussian curvature

Since the second fundamental form of the level set is $II = \frac{\nabla^2 u|_\Sigma}{|\nabla u|}$, then assume $\nu = \frac{\nabla u}{|\nabla u|}$, we have

$$|II|^2 = |\nabla u|^{-2}[|\nabla^2 u|^2 - 2|\nabla|\nabla u||^2 + (\nabla^2_{\nu\nu} u)^2],\ H^2 = |\nabla u|^{-2}(\Delta u - \nabla^2_{\nu\nu} u)^2. \tag{2.522}$$

In the hyperbolic region, $\mathrm{Ric}_{g_0} = -2g_0$, $R_{g_0} = -6$, then

$$R_\Sigma = R_{g_0} - 2\mathrm{Ric}_{g_0}(\nu,\nu) - |II|^2 + H^2 \tag{2.523}$$

$$= -6 + 4 - |\nabla u|^{-2}\left[|\nabla^2 u|^2 - 2|\nabla|\nabla u||^2 + (\nabla^2_{\nu\nu} u)^2\right] + |\nabla u|^{-2}(-3|\nabla u| - \nabla^2_{\nu\nu} u)^2 \tag{2.524}$$

$$= 7 + 6|\nabla u|^{-1}\nabla^2_{\nu\nu} u - |\nabla u|^{-2}|\nabla^2 u|^2 + 2|\nabla u|^{-2}|\nabla|\nabla u||^2. \tag{2.525}$$

If $u = v + w$, $w \in O_2(|v|^{3/2} r^{-l})$, then $|\nabla u| = |v|(1 + O_1(|v|^{1/2} r^{-l}))$,

$$|\nabla^2 u| = |\nabla^2 v| + O(|v|^{3/2} r^{-l}) = |v|(3 + O(|v|^{1/2} r^{-l})), \tag{2.526}$$

$$\nabla^2_{\nu\nu} u = \nabla^2_{\nu\nu} v + \nabla^2_{\nu\nu} w = -|v| + O(|v|^{3/2} r^{-l}), \tag{2.527}$$

$$|\nabla|\nabla u|| = |\nabla^2 u(\nu,\cdot)| = |\nabla^2 v(\nu,\cdot) + O(|v|^{3/2}r^{-l})| = |v| + O(|v|^{3/2}r^{-l}). \tag{2.528}$$

Therefore, $R_\Sigma = O(|v|^{1/2}r^{-l})$. Since we have an estimate for r as bellow:

$$\sqrt{r^2+1} = \frac{\rho^2 + x_3^2 + 4}{4x_3} \geq \frac{\rho^2+4}{4x_3} \geq C_1|u|(\rho^2+4),$$

then $|R_\Sigma| \leq C_2|u|^{-l+\frac{1}{2}}(\rho^2+4)^{-l}$.

Since $u_3 = 2x_3^{-2}(1 + O_1(|v|^{1/2}r^{-l}))$, according to line (2.510), $\sqrt{\det\tilde{g}} = x_3^{-3}u_3^{-1} = O(|u|)$,

$$|\nabla u| = |v|(1 + O_1(|v|^{1/2}r^{-l})) = |u|(1 + O_1(|v|^{1/2}r^{-l})), \tag{2.529}$$

then in the region $T_1 = \Omega \cap \{-\frac{2}{u} \leq \varepsilon + R^{-\delta}\}$, we have $D_{x_1,x_2} = \{x_1^2 + x_2^2 \leq 4R^2 - (\frac{2}{u} + 2R + \varepsilon)^2\}$ is the level set $\Sigma_u \cap \Omega$ parameterized by $\{x_1, x_2\}$.

$$\left|\int_{T_1} R_\Sigma |\nabla u| dV\right| \tag{2.530}$$

$$\leq C_2 \int_{T_1} |u|^{-l+\frac{1}{2}}(\rho^2+4)^{-l}|\nabla u| dV \tag{2.531}$$

$$= C_2 \int_{-\frac{2}{\varepsilon}}^{-\frac{2}{\varepsilon+R^{-\delta}}} \int_{D_{x_1,x_2}} |u|^{-l+\frac{1}{2}}(\rho^2+4)^{-l}|\nabla u|\sqrt{\det\tilde{g}}\,dx_1dx_2du \tag{2.532}$$

$$\leq C_3 \int_{-\frac{2}{\varepsilon}}^{-\frac{2}{\varepsilon+R^{-\delta}}} \int_{D_{x_1,x_2}} (\rho^2+4)^{-l}|u|^{\frac{5}{2}-l}dx_1dx_2du \tag{2.533}$$

$$= C_3 \int_{-\frac{2}{\varepsilon}}^{-\frac{2}{\varepsilon+R^{-\delta}}} \int_0^{\sqrt{4R^2-(\frac{2}{u}+2R+\varepsilon)^2}} \int_0^{2\pi} (\rho^2+4)^{-l}|u|^{\frac{5}{2}-l}\rho\, d\theta d\rho du \tag{2.534}$$

$$= 2\pi C_3 \int_{-\frac{2}{\varepsilon}}^{-\frac{2}{\varepsilon+R^{-\delta}}} |u|^{\frac{5}{2}-l} \int_0^{4R^2-(\frac{2}{u}+2R+\varepsilon)^2} (\rho^2+4)^{-l}\cdot\frac{1}{2}d(\rho^2)du \tag{2.535}$$

$$\leq C_4 \int_{-\frac{2}{\varepsilon}}^{-\frac{2}{\varepsilon+R^{-\delta}}} |u|^{\frac{5}{2}-l}du \tag{2.536}$$

$$\leq C_{5,\varepsilon}R^{-\delta}, \tag{2.537}$$

where the last inequality requires $l > 1$.

Hence, for any $\delta > 0$, when $R \to \infty$, the integral converges to 0.

In the region $T_2 = \Omega \cap \{-\frac{2}{u} \geq R^\delta\}$, similar to the calculation above, we have

$$\left|\int_{T_2} R_\Sigma |\nabla u| dV\right| \tag{2.538}$$

$$\leq 2\pi C_3 \int_{-\frac{2}{R^\delta}}^{-\frac{2}{4R+\varepsilon}} \int_0^{\sqrt{4R^2-(\frac{2}{u}+2R+\varepsilon)^2}} (\rho^2+4)^{-l}|u|^{\frac{5}{2}-l}\rho d\rho du \tag{2.539}$$

$$\leq C_6 \int_{-\frac{2}{R^\delta}}^{-\frac{2}{4R+\varepsilon}} |u|^{\frac{5}{2}-l} du \longrightarrow 0 \tag{2.540}$$

2.6.2 The boundary term $\partial_{\mathbf{n}}(|\nabla u| + u)$

In the section, we will estimate:

$$\int_{\partial\Omega} \partial_{\mathbf{n}}|\nabla u| + \langle \nabla u, \mathbf{n}\rangle dA = \int_{\partial\Omega} \langle \nabla(|\nabla u| + u), \mathbf{n}\rangle dA, \tag{2.541}$$

where $\mathbf{n}$ is the outer normal vector on $\partial\Omega$.

Since $\partial\Omega = \{x_1^2 + x_2^2 + (z - 2R - \varepsilon)^2 = 4R^2\}$, in the lower hemisphere, we have $(x_1, x_2, z) = (\rho\cos\varphi, \rho\sin\varphi, 2R + \varepsilon - \sqrt{4R^2 - \rho^2})$.

In the $\{x_1, x_2, z\}$ coordinate, $z = -\frac{2}{u}$.

Since $|\nabla\psi| = O(r^{-1})$, then

$$x_3 \frac{\partial z}{\partial x_3} = 2u^{-2} x_3 \frac{\partial u}{\partial x_3} \tag{2.542}$$

$$= -2u^{-2}\langle \frac{\nabla|v|}{|v|}, \nabla u\rangle \tag{2.543}$$

$$= -\frac{1}{2}x_3^2(1 + O(r^{-1}))[v + O(|v|r^{-1})] \tag{2.544}$$

$$= x_3(1 + O(r^{-1})). \tag{2.545}$$

As we have $|\nabla |v|^{\frac{3}{2}} r^{-\frac{3}{2}}| = O(|v| r^{-2})$, then

$$x_3 \frac{\partial z}{\partial x_1} = 2u^{-2} x_3 \frac{\partial u}{\partial x_1} \tag{2.546}$$

$$= 2u^{-2} O(|\nabla w|) \tag{2.547}$$

$$= 2u^{-2} O(|\nabla |v|^{\frac{3}{2}} r^{-\frac{3}{2}}| \cdot |\psi| + |v|^{\frac{3}{2}} r^{-\frac{3}{2}} |\nabla \psi|) \tag{2.548}$$

$$= 2|v|^{-2} O(|v| r^{-2} + |v|^{\frac{3}{2}} r^{-\frac{5}{2}}), \tag{2.549}$$

$$= O(|v|^{-1} r^{-2}). \tag{2.550}$$

Hence, $z = x_3(1 + O(r^{-1}))$, $\partial_3 z = 1 + O(r^{-1})$, $\partial_1 z = O(r^{-2})$, $\partial_2 z = O(r^{-2})$.

Since

$$dz = \frac{\partial z}{\partial x_1} dx_1 + \frac{\partial z}{\partial x_2} dx_2 + \frac{\partial z}{\partial x_3} dx_3, \tag{2.551}$$

then

$$g = \frac{1}{x_3^2}(dx_1^2 + dx_2^2 + dx_3^2) \tag{2.552}$$

$$= \frac{1}{x_3^2}\left[dx_1^2 + dx_2^2 + (\frac{\partial z}{\partial x_3})^{-2}(dz - \frac{\partial z}{\partial x_1} dx_1 - \frac{\partial z}{\partial x_2} dx_2)^2\right] \tag{2.553}$$

$$= \frac{1}{x_3^2}\Big[(1 + O(r^{-4}))dx_1^2 + (1 + O(r^{-4}))dx_2^2 + O(r^{-2}) dx_1 dz \tag{2.554}$$

$$+ O(r^{-2}) dx_2 dz + O(r^{-4}) dx_1 dx_2 + (1 + O(r^{-1}))dz^2\Big], \tag{2.555}$$

then on $\partial\Omega$,

$$dx_1 = \cos\varphi d\rho - \rho \sin\varphi d\varphi \tag{2.556}$$

$$dx_2 = \sin\varphi d\rho + \rho \cos\varphi d\varphi \tag{2.557}$$

$$dz = \frac{\rho}{\sqrt{4R^2 - \rho^2}} d\rho. \tag{2.558}$$

Hence, on $\partial\Omega$, in the $\{\rho, \varphi\}$ coordinate, we have the metric:

$$\hat{g} = \frac{1}{x_3^2}\Big[(1+O(r^{-4}))d\rho^2 + (1+O(r^{-4}))\rho^2 d\varphi^2 + O(r^{-4})\rho d\rho d\varphi \tag{2.559}$$

$$+ O(r^{-2})(\frac{\rho}{\sqrt{4R^2-\rho^2}}d\rho^2 + \frac{\rho^2}{\sqrt{4R^2-\rho^2}}d\rho d\varphi) + O(r^{-4})(d\rho^2 + \rho^2 d\varphi^2 + \rho d\rho d\varphi) \tag{2.560}$$

$$+ \frac{\rho^2}{4R^2-\rho^2}(1+O(r^{-1}))d\rho^2\Big] \tag{2.561}$$

$$= \frac{1}{x_3^2}\Big[\frac{4R^2}{4R^2-\rho^2}(1+O(r^{-1}))d\rho^2 + O(\frac{r^{-2}\rho^2}{\sqrt{4R^2-\rho^2}} + \rho r^{-4})d\rho d\varphi \tag{2.562}$$

$$+ (1+O(r^{-4}))\rho^2 d\varphi^2\Big], \tag{2.563}$$

then

$$\det\hat{g} = x_3^{-4}\Big[\frac{4R^2\rho^2}{4R^2-\rho^2}(1+O(r^{-1})) + O(\frac{r^{-4}\rho^4}{4R^2-\rho^2} + \rho^2 r^{-8})\Big] \tag{2.564}$$

$$= x_3^{-4}\frac{4R^2\rho^2}{4R^2-\rho^2}(1+O(r^{-1})). \tag{2.565}$$

In the rest of this section, we divide the region $\partial\Omega$ into several pieces to apply different estimates.

The region $\rho > R^\delta$

If $\rho > R^\delta$, $\partial\Omega^-$, $\partial\Omega^+$ are lower and upper hemispheres.

Since $|\nabla w| = O(|v|r^{-2})$, $|\nabla^2 w| = O(|v|r^{-2})$, then

$$
\begin{aligned}
&|\nabla(|\nabla u| + u)| && (2.566)\\
=&\left|\nabla^2 u\left(\frac{\nabla u}{|\nabla u|}, \cdot\right) + \nabla u\right| && (2.567)\\
=&\left|\nabla^2 v\left(\frac{\nabla u}{|\nabla u|}, \cdot\right) + \nabla v\right| + O(|v|r^{-2}) && (2.568)\\
=&\left|\frac{v\nabla u}{|\nabla u|} + \nabla v\right| + O(|v|r^{-2}) && (2.569)\\
=&|\nabla u|^{-1}\Big|v\nabla w + (|\nabla u| - |\nabla v|)\nabla v\Big| + O(|v|r^{-2}) && (2.570)\\
=&O(|v|r^{-2}). && (2.571)
\end{aligned}
$$

Hence, we can estimate the boundary integral as below

$$
\begin{aligned}
&\int_{\partial\Omega^- \cap \{\rho > R^\delta\}} \nabla_X(|\nabla u| + u)dA && (2.572)\\
=&\int_{R^\delta}^{2R}\int_0^{2\pi} O(|v|r^{-2})\sqrt{\det \hat{g}}d\varphi d\rho && (2.573)\\
\leq& C\int_{R^\delta}^{2R}\int_0^{2\pi} r^{-2}|v|x_3^{-2}\rho\frac{2R}{\sqrt{4R^2 - \rho^2}}d\varphi d\rho && (2.574)\\
\leq& C\int_{R^\delta}^{2R}\int_0^{2\pi} (\rho^2 + 4)^{-2}x_3^{-1}\rho\frac{2R}{\sqrt{4R^2 - \rho^2}}d\varphi d\rho && (2.575)
\end{aligned}
$$

In line (2.575), we use $t = \frac{\rho^2 + x_3^2 + 4}{4x_3} \geq \frac{1}{4}(\rho^2 + 4)x_3^{-1}$. Since

$$
(z - \varepsilon)(4R + \varepsilon - z) = 4R^2 - (z - 2R - \varepsilon)^2 = \rho^2, \tag{2.576}
$$

then $\varepsilon + \frac{\rho^2}{4R} \leq z \leq \varepsilon + R^{-1}\rho^2$.

When $R \leq \rho \leq 2R$, $\varepsilon + \frac{R}{4} \leq z \leq 4R + \varepsilon$, then $c_1 R \leq x_3 \leq c_2 R$,

$$\int_R^{2R} \int_0^{2\pi} (\rho^2 + 4)^{-2} x_3^{-1} \rho \frac{2R}{\sqrt{4R^2 - \rho^2}} d\varphi d\rho \tag{2.577}$$

$$\leq C \int_R^{2R} \rho^{-3} R^{-1} \frac{2R}{\sqrt{4R^2 - \rho^2}} d\rho \tag{2.578}$$

$$\leq C \int_R^{2R} \frac{R^{-1}}{\sqrt{4R^2 - \rho^2}} d\rho \to 0 \tag{2.579}$$

When $R^\delta \leq \rho \leq R$, since $z = x_3(1 + O(r^{-1}))$, then $c_1(\varepsilon + \frac{\rho^2}{4R}) \leq x_3 \leq c_2(\varepsilon + R^{-1}\rho^2)$,

$$\int_{R^\delta}^{R} \int_0^{2\pi} (\rho^2 + 4)^{-2} x_3^{-1} \rho \frac{2R}{\sqrt{4R^2 - \rho^2}} d\varphi d\rho \tag{2.580}$$

$$\leq C \int_{R^\delta}^{R} \int_0^{2\pi} \rho^{-3} x_3^{-1} d\varphi d\rho \tag{2.581}$$

$$\leq C \int_{R^\delta}^{R} \int_0^{2\pi} \rho^{-3} (c_1 \varepsilon)^{-1} d\varphi d\rho \to 0 \tag{2.582}$$

Since the area of upper hemisphere is finite, then the boundary integral converges to 0, when $R \to \infty$.

The region $\delta_0^{-1} \leq \rho \leq R^\delta$

Let $F = x_1^2 + x_2^2 + (z - 2R - \varepsilon)^2$, then we have the outer normal vector $\frac{\nabla F}{|\nabla F|}$. To estimate the boundary integral more precisely in this region, we need to estimate the outer normal vector.

Lemma 45. *The unit outer normal vector is close to $\frac{\nabla |v|}{|v|}$.*

$$\frac{\nabla F}{|\nabla F|} = (1 + O(r^{-1})) \frac{\nabla |v|}{|v|} + O(r^{-2}). \tag{2.583}$$

Proof. As $z = x_3(1 + O(r^{-1}))$, $\partial_3 z = 1 + O(r^{-1})$, $\partial_1 z = O(x_3^{1/2} r^{-3/2})$, $\partial_2 z =$

$O(x_3^{1/2}r^{-3/2})$, then

$$
\begin{aligned}
\frac{1}{2}dF =& x_1dx_1 + x_2dx_2 + (z-2R-\varepsilon)dz && (2.584)\\
=& 2x_3dt - (\frac{x_3}{2} - \frac{x_1^2+x_2^2+4}{2x_3})dx_3 + (z-2R-\varepsilon)dz && (2.585)\\
=& 2x_3dt + (2t-x_3)dx_3 + (z-2R-\varepsilon)dz && (2.586)\\
=& 2x_3dt + (2t-x_3)dx_3 + (z-2R-\varepsilon)(\frac{\partial z}{\partial x_3}dx_3 + \frac{\partial z}{\partial x_1}dx_1 + \frac{\partial z}{\partial x_2}dx_2) && (2.587)\\
=& 2x_3dt + (2t-x_3)dx_3 + (z-2R-\varepsilon)(1+O(r^{-1}))dx_3 + O(Rx_3r^{-2}) && (2.588)\\
=& 2x_3dt + (2t-2R-\varepsilon)dx_3 + O(Rr^{-1}+x_3r^{-1})dx_3 + O(Rx_3r^{-2}), && (2.589)
\end{aligned}
$$

$$|2x_3dt| = 2x_3r \leq 8R^{2\delta} \leq Rx_3r^{-2}, \quad (2.590)$$

$$|(2t-\varepsilon)dx_3| = x_3(2t-\varepsilon) \leq 8R^{2\delta} \leq Rx_3r^{-2}, \quad (2.591)$$

therefore,

$$\frac{1}{2}dF = -2R(1+O(r^{-1}))dx_3 + O(Rx_3r^{-2}). \quad (2.592)$$

Then we have

$$\left|\frac{1}{2}dF\right| = 2Rx_3(1+O(r^{-1})), \quad (2.593)$$

$$\frac{dF}{|dF|} = -x_3^{-1}(1+O(r^{-1}))dx_3 + O(r^{-2}) = (1+O(r^{-1}))|v|^{-1}d|v| + O(r^{-2}), \quad (2.594)$$

therefore,

$$\frac{\nabla F}{|\nabla F|} = (1+O(r^{-1}))\frac{\nabla|v|}{|v|} + O(r^{-2}). \quad (2.595)$$

□

We have $|\nabla\psi| = O(r^{-1})$, $|\nabla^2\psi| = O(r^{-2+\varepsilon})$, then

$$|\nabla u| = |v| - \frac{3}{2}|v|^{\frac{1}{2}}t^{-\frac{5}{2}}\psi - |v|^{\frac{3}{2}}t^{-\frac{3}{2}}\frac{\nabla|v|}{|v|}\cdot\nabla\psi + O_1(|v|t^{-4}), \tag{2.596}$$

therefore,

$$|\nabla u| + u = |v|^{\frac{3}{2}}t^{-\frac{3}{2}}\psi - |v|^{\frac{3}{2}}t^{-\frac{3}{2}}\frac{\nabla|v|}{|v|}\cdot\nabla\psi + O_1(|v|t^{-4} + |v|^{\frac{1}{2}}t^{-\frac{5}{2}}). \tag{2.597}$$

From Corollary 31, $|\nabla\psi| = O(r^{-1})$ and $|\nabla^2\psi| = O(r^{-2+\varepsilon})$, then $|\nabla(|\nabla u| + u)| = O(|r^{-1}|)$.

Since

$$\langle\frac{\nabla|v|}{|v|}, \nabla(|v|^{\frac{3}{2}}t^{-\frac{3}{2}})\rangle = \frac{3}{2}|v|^{\frac{1}{2}}t^{-\frac{5}{2}}, \tag{2.598}$$

then let us estimate the main part of the integrand

$$\langle\frac{\nabla|v|}{|v|}, \nabla(|\nabla u| + u)\rangle \tag{2.599}$$

$$= \frac{3}{2}|v|^{\frac{1}{2}}t^{-\frac{5}{2}}\psi + |v|^{\frac{3}{2}}t^{-\frac{3}{2}}\frac{\nabla|v|}{|v|}\cdot\nabla\psi - \frac{3}{2}|v|^{\frac{1}{2}}t^{-\frac{5}{2}}\frac{\nabla|v|}{|v|}\cdot\nabla\psi \tag{2.600}$$

$$- |v|^{\frac{3}{2}}t^{-\frac{3}{2}}\nabla\left(\frac{\nabla|v|}{|v|}\right)(\nabla\psi, \frac{\nabla|v|}{|v|}) - |v|^{\frac{3}{2}}t^{-\frac{3}{2}}\nabla^2\psi(\frac{\nabla|v|}{|v|}, \frac{\nabla|v|}{|v|}) + O(|v|^{\frac{1}{2}}t^{-\frac{5}{2}}), \tag{2.601}$$

and the error term

$$\langle\frac{\nabla F}{|\nabla F|} - \frac{\nabla|v|}{|v|}, \nabla(|\nabla u| + u)\rangle \tag{2.602}$$

$$= \langle O(r^{-1})\frac{\nabla|v|}{|v|} + O(r^{-2}), O(r^{-1})\rangle\rangle \tag{2.603}$$

$$= O(r^{-2}). \tag{2.604}$$

We need to estimate the second term of line (2.600) using $\psi = A + \tilde{\psi}$ and $|\nabla^{S^2}\psi| = O(1)$,

$$\left|\frac{\nabla|v|}{|v|}\cdot\nabla\psi\right| = \left|\langle(\frac{1}{r} - \frac{1}{rt|v|})\nabla r - \frac{r\sin\theta}{|v|}\nabla\theta, \nabla\psi\rangle\right| \tag{2.605}$$

$$\leq \left|\frac{\nabla r}{r}\cdot\nabla\tilde{\psi}\right| + \frac{r^{-1}|\sin\theta|}{|v|}|\nabla^{S^2}\psi| \tag{2.606}$$

$$\leq O(r^{-2}) + O(|v|^{-\frac{1}{2}}r^{-\frac{3}{2}}). \tag{2.607}$$

For the first term in (2.601), we have

$$\nabla\left(\frac{\nabla|v|}{|v|}\right)(\nabla\psi, \frac{\nabla|v|}{|v|}) \tag{2.608}$$

$$= \frac{\nabla^2|v|}{|v|}(\nabla\psi, \frac{\nabla|v|}{|v|}) - \frac{\nabla|v|\otimes\nabla|v|}{|v|^2}(\nabla\psi, \frac{\nabla|v|}{|v|}) \tag{2.609}$$

$$= \nabla\psi\cdot\frac{\nabla|v|}{|v|} - \frac{\nabla|v|}{|v|}\cdot\nabla\psi \tag{2.610}$$

$$= 0. \tag{2.611}$$

Therefore, we have

$$\langle\frac{\nabla F}{|\nabla F|}, \nabla(|\nabla u| + u)\rangle = -|v|^{\frac{3}{2}}t^{-\frac{3}{2}}\nabla^2\psi\left(\frac{\nabla|v|}{|v|}, \frac{\nabla|v|}{|v|}\right) + O(r^{-2}). \tag{2.612}$$

Since $\left||v|^{-1}\nabla|v| - t^{-1}\nabla t\right|^2 = 2|v|^{-1}t^{-1}$, and $|\nabla^2\psi| = O(r^{-2+\varepsilon})$, then

$$\nabla^2\psi\left(\frac{\nabla|v|}{|v|}, \frac{\nabla|v|}{|v|}\right) \tag{2.613}$$

$$= \nabla^2\psi(t^{-1}\nabla t, t^{-1}\nabla t) + O(|v|^{-\frac{1}{2}}r^{-\frac{5}{2}+\varepsilon}) \tag{2.614}$$

$$= \nabla^2\psi(r\partial_r, r\partial_r) + O(|v|^{-\frac{1}{2}}r^{-\frac{5}{2}+\varepsilon}) \tag{2.615}$$

$$= r^2\partial_r^2\psi - r^3t^{-2}\partial_r\psi + O(|v|^{-\frac{1}{2}}r^{-\frac{5}{2}+\varepsilon}), \tag{2.616}$$

for the last equality, we use $\Gamma^r_{rr} = -rt^{-2}$.

We need C^0 estimates: $|\partial_r^2\tilde{\psi}| = O(r^{-4})$, $|\partial_r\tilde{\psi}| = O(r^{-3})$. However, we can not get such C^0 estimates. To solve this issue, we expand $\tilde{\psi} = Br^{-2} + \hat{\psi}$, where B and $\hat{\psi}$ satisfy

$$B \in H^{1-\varepsilon}(S^2), \qquad \sum_{j=1}^{\infty}\langle\hat{\psi}, \chi_j\rangle^2 = O(r^{-6+2\varepsilon}), \tag{2.617}$$

$$\sum_{j=1}^{\infty}\lambda_j^{-1}\langle\partial_r\hat{\psi}, \chi_j\rangle^2 = O(r^{-8+2\varepsilon}), \qquad \sum_{j=1}^{\infty}\lambda_j^{-2}\langle\partial_r^2\hat{\psi}, \chi_j\rangle^2 = O(r^{-10+2\varepsilon}) \tag{2.618}$$

To conclude the boundary integral converges to 0, we need to show

$$\int_0^{2\pi}\int_{\delta_0^{-1}}^{R^\delta}|v|^{\frac{3}{2}}t^{-\frac{3}{2}}\nabla^2\psi(t\partial r, t\partial r)\sqrt{\det\hat{g}}d\rho d\varphi \to 0. \tag{2.619}$$

We want to apply Cauchy-Schwartz inequality for Sobolev space, therefore, we need to transfer the $\{\rho, \varphi\}$ coordinate into $\{\theta, \varphi\}$ coordinate.

$$\rho = \sqrt{x_1^2 + x_2^2} = \frac{2r\sin\theta}{|v|} = \frac{2\sin\theta}{1+\cos\theta}(1 + O_1(r^{-2}|v|)), \tag{2.620}$$

and in this region, we have $c\varepsilon\delta_0^{-1} \le t \le C\varepsilon R^{2\delta}$, then

$$\begin{aligned}
dvol_{\hat{g}} =& x_3^{-2}\frac{2R\rho}{\sqrt{4R^2-\rho^2}}(1+O(r^{-1}))d\rho d\varphi && (2.621)\\
=& x_3^{-2}\frac{2R\rho}{\sqrt{4R^2-\rho^2}}(1+O(r^{-1}))\frac{\partial\rho}{\partial\theta}d\theta d\varphi && (2.622)\\
=& x_3^{-2}\rho(1+O(R^{-2}\rho^2))(1+O(r^{-1}))\cdot\frac{2}{1+\cos\theta}d\rho d\varphi && (2.623)\\
=& 4t^2(1+\cos\theta)^2\cdot\frac{4\sin\theta}{(1+\cos\theta)^2}(1+O(r^{-1}))d\theta d\varphi && (2.624)\\
=& 16t^2(1+O(r^{-1}))\sin\theta d\theta d\varphi. && (2.625)
\end{aligned}$$

Since $\tilde{\psi} = Br^{-2} + \hat{\psi}$, then

$$\nabla^2\psi\,(t\partial r, t\partial r) = t^2\nabla^2\tilde{\psi}\,(\partial r, \partial r) \tag{2.626}$$

$$= t^2\partial_r^2\tilde{\psi} - t^2\nabla_{\nabla_{\partial r}\partial r}\tilde{\psi} \tag{2.627}$$

$$= t^2\partial_r^2\tilde{\psi} + r\partial_r\tilde{\psi} \tag{2.628}$$

$$= (4r^2+6)r^{-4}B + t^2\partial_r^2\hat{\psi} + r\partial_r\hat{\psi}. \tag{2.629}$$

Since

$$\rho = \frac{2\sin\theta}{1+\cos\theta}(1+O_1(r^{-2}|v|)) = 4(\pi-\theta)^{-1}(1+O((\pi-\theta)^2))(1+O_1(r^{-2}|v|)), \tag{2.630}$$

then we can find $\delta_1 \le C\delta_0$ such that up to the change of coordinate, we have

$$\{(\rho,\varphi)|\delta_0^{-1} \le \rho \le R^\delta\} \subset \tilde{\Omega} := \{(\theta,\varphi)|\pi - \delta_1 \le \theta \le \pi, 0 \le \varphi \le 2\pi\}.$$

Let us apply the properties of $\hat{\psi}$ and B to estimate the integral

$$\int_0^{2\pi}\int_{\delta_0^{-1}}^{R^\delta} |v|^{\frac{3}{2}} t^{-\frac{3}{2}}\nabla^2\psi(t\partial r, t\partial r)\sqrt{\det\hat{g}}\,d\rho d\varphi \tag{2.631}$$

$$\le C\int_0^{2\pi}\int_{\pi-\delta_1}^{\pi} \nabla^2\psi(t\partial r, t\partial r)\cdot t^2(1+\cos\theta)^{\frac{3}{2}}\sin\theta d\theta d\varphi \tag{2.632}$$

$$\le C\int_0^{2\pi}\int_{\pi-\delta_1}^{\pi} (|B| + r^4|\partial_r^2\hat{\psi}| + r^3|\partial_r\hat{\psi}|)(1+\cos\theta)^{\frac{3}{2}}\sin\theta d\theta d\varphi \tag{2.633}$$

$$\le C\Bigg(\|B\|_{L^2(S^2)}\cdot\|(1+\cos\theta)^{\frac{3}{2}}\|_{L^2(\tilde{\Omega})} + \|r^4\partial_r^2\hat{\psi}\|_{H^{-2}}\cdot\|(1+\cos\theta)^2(\sin\theta)^{-1}\|_{H^2(\tilde{\Omega})} \tag{2.634}$$

$$+\|r^3\partial_r\hat{\psi}\|_{H^{-1}(\tilde{\Omega})}\cdot\|(1+\cos\theta)^2(\sin\theta)^{-1}\|_{H^1(\tilde{\Omega})}\Bigg). \tag{2.635}$$

Then if let $\delta_0 \to 0$, then $\delta_1 \to 0$, therefore, the measure of the region $\tilde{\Omega}$ on S^2 goes to 0.

Hence, we have the boundary integral in the region converges to 0.

The region $\rho \le \delta_0^{-1}$

In this section, we are away from $\theta = \pi$, then A and B are smooth. The C^k norms of A and B depend on δ_0. For convenience, we use the expansion $w = |v|r^{-1}(\tilde{A} + \tilde{B}r^{-2} + O(r^{-3}))$. Then we also have $\tilde{A}$ and $\tilde{B}$ are smooth, since

$$\tilde{A} = \sqrt{1+\cos\theta}A,\ \tilde{B} = \sqrt{1+\cos\theta}B + \left(\frac{1}{4\sqrt{1+\cos\theta}} - \frac{3}{4}\sqrt{1+\cos\theta}\right)A. \quad (2.636)$$

Recall that A and B satisfy $6B = \frac{15}{4}A - \Delta_{S^2}A$, then $\tilde{A}$ and $\tilde{B}$ satisfy

$$6\tilde{B} = -\tilde{A} - \Delta_{S^2}\tilde{A} + \frac{\tilde{A} - \sin\theta\partial_\theta\tilde{A}}{1+\cos\theta}. \quad (2.637)$$

If $\rho \le R^\delta$, and we only consider the lower hemisphere, then

$$(z-\varepsilon)(4R+\varepsilon-z) = 4R^2 - (z-2R-\varepsilon)^2 = \rho^2, \quad (2.638)$$

therefore, $z \le \varepsilon + \frac{1}{3}R^{2\delta-1}$, then $x_3 = z(1+O(r^{-1})) \le 2\varepsilon$ and $x_3 \ge \frac{1}{2}\varepsilon$.

$$t = \frac{x_1^2+x_2^2+x_3^2+4}{4x_3} \quad (2.639)$$

$$= \frac{z-\varepsilon}{4x_3}(4R+\varepsilon-z) + \frac{x_3}{4} + \frac{1}{x_3} \quad (2.640)$$

$$\le 2R^{2\delta}\varepsilon^{-1} \quad (2.641)$$

Since $t = \frac{x_1^2+x_2^2+x_3^2+4}{4x_3}$, then

$$dt = \frac{x_1}{2x_3}dx_1 + \frac{x_2}{2x_3}dx_2 + \left(\frac{1}{4} - \frac{x_1^2+x_2^2+4}{4x_3^2}\right)dx_3, \tag{2.642}$$

Since $1+\cos\theta = \frac{|v|}{r} - \frac{1}{2r^2} + O(r^{-3})$, then

$$
\begin{aligned}
&|\nabla u| + u && (2.643)\\
=&\frac{\tilde{A}(1+\cos\theta)}{r^2} - \frac{\tilde{A}}{2r^2} + \tilde{A}(1+\cos\theta) + \frac{3(1+\cos\theta)\tilde{B}}{r^2} + \frac{\sin\theta\partial_\theta\tilde{A}}{r^2} && (2.644)\\
&+O(\frac{1}{r^3} + \frac{1}{r^2|v|}) && (2.645)\\
=&\frac{\tilde{A}\left(\frac{|v|}{r} - \frac{1}{2r^2}\right)}{r^2} - \frac{\tilde{A}}{2r^2} + \tilde{A}(\frac{|v|}{r} - \frac{1}{2r^2}) + \frac{3|v|\tilde{B}}{r^3} + \frac{\nabla v\cdot\nabla\tilde{A}}{r} + O(\frac{1}{r^3} + \frac{1}{r^2|v|}) && (2.646)\\
=&\frac{\tilde{A}|v|}{r^3} - \frac{\tilde{A}}{r^2} + \frac{\tilde{A}|v|}{r} + \frac{3|v|\tilde{B}}{r^3} + \frac{\nabla v\cdot\nabla\tilde{A}}{r} + O(\frac{1}{r^3} + \frac{1}{r^2|v|}). && (2.647)
\end{aligned}
$$

Since $x_3 \le 2\varepsilon$, $|v| = \frac{2}{x_3} > \varepsilon^{-1}$, then

$$
\begin{aligned}
\langle\frac{\nabla|v|}{|v|}, \nabla r\rangle =&|v|^{-1}\langle(\frac{r}{\sqrt{r^2+1}} + \cos\theta)\nabla r - r\sin\theta\nabla\theta, \nabla r\rangle && (2.648)\\
=&r\sqrt{r^2+1}|v|^{-1} + (r^2+1)|v|^{-1}\cos\theta && (2.649)\\
=&(1+r^2)r^{-1} - \frac{\sqrt{1+r^2}}{r|v|} && (2.650)\\
=&r + O(1). && (2.651)
\end{aligned}
$$

Hence,

$$\langle\frac{\nabla|v|}{|v|},\nabla(|\nabla u|+u)\rangle \tag{2.652}$$
$$=\langle\frac{\nabla|v|}{|v|},\nabla\left[\frac{\tilde{A}|v|}{r^3}-\frac{\tilde{A}}{r^2}+\frac{\tilde{A}|v|}{r}+\frac{3|v|\tilde{B}}{r^3}+\frac{\nabla v\cdot\nabla\tilde{A}}{r}+O(\frac{1}{r^3}+\frac{1}{r^2|v|})\right]\rangle \tag{2.653}$$
$$=(\tilde{A}+3\tilde{B})|v|r^{-3}-3(\tilde{A}+3\tilde{B})|v|r^{-3}+2\tilde{A}r^{-2}+\tilde{A}|v|r^{-1} \tag{2.654}$$
$$-\tilde{A}|v|r^{-2}[(1+r^2)r^{-1}-\frac{\sqrt{1+r^2}}{r|v|}]+r^{-1}\nabla|v|\cdot\nabla\tilde{A}-r^{-1}\nabla v\cdot\nabla\tilde{A} \tag{2.655}$$
$$-r^{-1}\nabla|v|\cdot\nabla\tilde{A}-r^{-1}|v|^{-1}(\nabla^2\tilde{A})(\nabla|v|,\nabla|v|)+O(r^{-3}+r^{-2}|v|^{-1}) \tag{2.656}$$
$$=-3(\tilde{A}+2\tilde{B})|v|r^{-3}+3\tilde{A}r^{-2}+r^{-1}\nabla|v|\cdot\nabla\tilde{A}-r^{-1}|v|^{-1}(\nabla^2\tilde{A})(\nabla|v|,\nabla|v|) \tag{2.657}$$
$$+O(r^{-3}+r^{-2}|v|^{-1}). \tag{2.658}$$

Here are the non-vanishing Christoffel symbols on hyperbolic space:

$$\Gamma^r_{rr}=-\frac{r}{1+r^2},\ \Gamma^\theta_{r\theta}=\Gamma^\theta_{\theta r}=\Gamma^\varphi_{r\varphi}=\Gamma^\varphi_{\varphi r}=\frac{1}{r},\ \Gamma^r_{\theta\theta}=-r(1+r^2),$$

$$\Gamma^\varphi_{\theta\varphi}=\Gamma^\varphi_{\varphi\theta}=\cot\theta,\ \Gamma^r_{\varphi\varphi}=-r(1+r^2)\sin^2\theta,\ \Gamma^\theta_{\varphi\varphi}=-\cos\theta\sin\theta.$$

Then we have:

$$(\nabla^2\tilde{A})(\nabla|v|,\nabla|v|)=-2\Gamma^\theta_{r\theta}\partial_\theta\tilde{A}g^{rr}\partial_r|v|g^{\theta\theta}\partial_\theta|v|+O(r^{-2}) \tag{2.659}$$
$$=2r^{-1}(\partial_\theta\tilde{A})(1+r^2)(\frac{r}{\sqrt{1+r^2}}+\cos\theta)\cdot r^{-2}\cdot r\sin\theta+O(r^{-2}) \tag{2.660}$$
$$=-2|v|\nabla|v|\cdot\nabla\tilde{A}+O(r^{-1}) \tag{2.661}$$

$$6\tilde{B} = -\tilde{A} - \Delta_{S^2}\tilde{A} + \frac{\tilde{A} - \sin\theta\partial_\theta\tilde{A}}{1+\cos\theta} \tag{2.662}$$

$$= -\tilde{A} - r^2\Delta\tilde{A} + \frac{r\tilde{A} + r^2\nabla|v|\cdot\nabla\tilde{A}}{|v|} + O(r^{-1}). \tag{2.663}$$

Hence,

$$r^2\langle\frac{\nabla|v|}{|v|}, \nabla(|\nabla u| + u)\rangle \tag{2.664}$$

$$= -3(\tilde{A} + 2\tilde{B})|v|r^{-1} + 3\tilde{A} + r\nabla|v|\cdot\nabla\tilde{A} - r|v|^{-1}(\nabla^2\tilde{A})(\nabla|v|, \nabla|v|) \tag{2.665}$$

$$+ O(r^{-1} + |v|^{-1}) \tag{2.666}$$

$$= -2\tilde{A}|v|r^{-1} + |v|r\Delta\tilde{A} + 2\tilde{A} + 2r\nabla|v|\cdot\nabla\tilde{A} + O(|v|^{-1}) \tag{2.667}$$

$$= -2\tilde{A}(1+\cos\theta) + (1+\cos\theta)\left(\partial_{\theta\theta}\tilde{A} + \frac{\cos\theta}{\sin\theta}\partial_\theta\tilde{A}\right) + 2\tilde{A} \tag{2.668}$$

$$-2\sin\theta\partial_\theta\tilde{A} + O(|v|^{-1}) \tag{2.669}$$

$$=(1+\cos\theta)\partial_{\theta\theta}\tilde{A} + \left[(1+\cos\theta)\frac{\cos\theta}{\sin\theta} - 2\sin\theta\right]\partial_\theta\tilde{A} - 2\tilde{A}\cos\theta + O(|v|^{-1}) \tag{2.670}$$

Currently, it is difficult to see whether the integral of line (2.667) converges to 0.

Since we will integrate line (2.667) with respect to[3] φ and ρ, then we can assume $\tilde{A}$ only depends on θ. Suppose $\tilde{A} = f(\cos\theta)$. Assume $\cos\theta = s$. Since $\tilde{A}$ is smooth on S^2, except for $\theta = \pi$, we have $f'(s)$, $f''(s)$ and $f'''(s)$ are bounded, $s \in [-1, 1]$. Replacing $\tilde{A}$ by $f(\cos\theta)$, then we simplify line (2.670) into line (2.672) and (2.673).

$$r^2\langle\frac{\nabla|v|}{|v|}, \nabla(|\nabla u| + u)\rangle \tag{2.671}$$

$$=(1+\cos\theta)\left[\sin^2\theta f''(\cos\theta) - \cos\theta f'(\cos\theta)\right] - 2\cos\theta f(\cos\theta) \tag{2.672}$$

$$-(\cos\theta + \cos^2\theta - 2\sin^2\theta)f'(\cos\theta) + O(|v|^{-1}) \tag{2.673}$$

$$=(1+s)^2(1-s)f''(s) - (2s + 4s^2 - 2)f'(s) - 2sf(s) + O(|v|^{-1}). \tag{2.674}$$

[3]The φ in the $\{\rho, \varphi\}$ is the same as the φ in $\{r, \theta, \varphi\}$.

From line (2.595), assume $X = \frac{\nabla F}{|\nabla F|}$, we have: $X = (1 + O(r^{-1}))\frac{\nabla|v|}{|v|} + O(x_3^{1/2} r^{-3/2})$, since

$$|\nabla(|\nabla u| + u)| = O(r^{-1}), \tag{2.675}$$

$$\langle \frac{\nabla|v|}{|v|}, \nabla(|\nabla u| + u)\rangle = O(r^{-2}), \tag{2.676}$$

then

$$\int_{\partial\Omega^- \cap \{\rho \leq R^\delta\}} \nabla_X(|\nabla u| + u) dA \tag{2.677}$$

$$= \int_{\partial\Omega^- \cap \{\rho \leq R^\delta\}} \langle (1 + O(r^{-1}))\frac{\nabla|v|}{|v|}, \nabla(|\nabla u| + u)\rangle + O(r^{-5/2} x_3^{1/2}) dA \tag{2.678}$$

$$= \int_0^{R^\delta} \int_0^{2\pi} \left[\langle \frac{\nabla|v|}{|v|}, \nabla(|\nabla u| + u)\rangle + O(r^{-3} + r^{-5/2} x_3^{1/2}) \right] \sqrt{\det \hat{g}} d\varphi d\rho \tag{2.679}$$

$$= \int_0^{R^\delta} \int_0^{2\pi} \left[\langle \frac{\nabla|v|}{|v|}, \nabla(|\nabla u| + u)\rangle + O(r^{-5/2} x_3^{1/2}) \right] x_3^{-2} \frac{2R\rho}{\sqrt{4R^2 - \rho^2}} (1 + O(r^{-1})) d\varphi d\rho \tag{2.680}$$

$$= \int_0^{R^\delta} \int_0^{2\pi} \langle \frac{\nabla|v|}{|v|}, \nabla(|\nabla u| + u)\rangle x_3^{-2} \frac{2R\rho}{\sqrt{4R^2 - \rho^2}} + O(r^{-5/2} x_3^{-3/2} \rho + r^{-3} x_3^{-2} \rho) d\varphi d\rho \tag{2.681}$$

$$= \int_0^{R^\delta} \int_0^{2\pi} \langle \frac{\nabla|v|}{|v|}, \nabla(|\nabla u| + u)\rangle x_3^{-2} \frac{2R\rho}{\sqrt{4R^2 - \rho^2}} + O((\rho^2 + 4)^{-5/2} \rho x_3) d\varphi d\rho \tag{2.682}$$

$$= \int_0^{R^\delta} \int_0^{2\pi} \langle \frac{\nabla|v|}{|v|}, \nabla(|\nabla u| + u)\rangle x_3^{-2} \frac{2R\rho}{\sqrt{4R^2 - \rho^2}} d\varphi d\rho + O(\varepsilon) \tag{2.683}$$

$$= \int_0^{R^\delta} \int_0^{2\pi} r^2 \langle \frac{\nabla|v|}{|v|}, \nabla(|\nabla u| + u)\rangle r^{-2} x_3^{-2} \rho d\varphi d\rho + O(R^{-2+2\delta} + \varepsilon) \tag{2.684}$$

Since $\sqrt{r^2+1} = \frac{\rho^2+x_3^2+4}{4x_3}$, then

$$r^{-2}x_3^{-2}\rho = \frac{16\rho}{(\rho^2+x_3^2+4)^2}(1+O(r^{-2})) \tag{2.685}$$
$$= \frac{16\rho}{(\rho^2+4)^2}(1+O(\varepsilon^2))(1+O(r^{-2})) \tag{2.686}$$
$$= \frac{16\rho}{(\rho^2+4)^2}(1+O(\varepsilon^2)) \tag{2.687}$$

and we also have

$$\cos\theta = \frac{|v|}{r} - 1 + O(r^{-2}) \tag{2.688}$$
$$= \frac{2}{x_3 t} - 1 + O(r^{-2}) \tag{2.689}$$
$$= \frac{8}{\rho^2+x_3^2+4} - 1 + O(r^{-2}) \tag{2.690}$$
$$= \frac{8}{\rho^2+4} - 1 + O(r^{-2} + x_3^2(\rho^2+4)^{-2}) \tag{2.691}$$
$$= \frac{8}{\rho^2+4} - 1 + O(\varepsilon^2), \tag{2.692}$$

assume $\tilde{s} = \frac{8}{\rho^2+4} - 1$, since $s = \cos\theta$, then $|s-\tilde{s}| = O(\varepsilon^2)$ implies

$$|f''(\tilde{s}) - f''(s)| + |f'(\tilde{s}) - f'(s)| + |f(\tilde{s}) - f(s)| \le C_f\varepsilon^2, \tag{2.693}$$

where C_f depends on $||f||_{C^3}$.

Combining (2.674) , (2.684), (2.687), and (2.693), we have

$$\int_{\partial\Omega^- \cap \{\rho \le R^\delta\}} \nabla_X(|\nabla u| + u) \tag{2.694}$$

$$= \int_0^{R^\delta} \int_0^{2\pi} \Big[(1+s)^2(1-s)f''(s) - (2s+4s^2-2)f'(s) - 2sf(s) \tag{2.695}$$

$$+ O(|v|^{-1}) \Big] \frac{16\rho}{(\rho^2+4)^2}(1+O(\varepsilon^2))d\varphi d\rho + O(R^{-2+2\delta} + \varepsilon) \tag{2.696}$$

$$= \int_1^{\frac{8}{R^{2\delta}+4}-1} \int_0^{2\pi} \Big[(1+s)^2(1-s)f''(s) - (2s+4s^2-2)f'(s) - 2sf(s) \Big] d\varphi d\tilde{s} \tag{2.697}$$

$$+ O(R^{-2+2\delta} + \varepsilon) \tag{2.698}$$

$$= \int_1^{-1} \int_0^{2\pi} \Big[(1+s)^2(1-s)f''(s) - (2s+4s^2-2)f'(s) - 2sf(s) \Big] d\varphi d\tilde{s} \tag{2.699}$$

$$+ O(R^{-2+2\delta} + R^{-2\delta} + \varepsilon) \tag{2.700}$$

$$=2\pi \int_1^{-1} \Big[(1+\tilde{s})^2(1-\tilde{s})f''(\tilde{s}) - (2\tilde{s}+4\tilde{s}^2-2)f'(\tilde{s}) - 2\tilde{s}f(\tilde{s}) \Big] d\tilde{s} \tag{2.701}$$

$$+ O(R^{-2\delta} + \varepsilon) \tag{2.702}$$

$$=2\pi f'(\tilde{s})(1+\tilde{s})^2(1-\tilde{s}) \Big|_1^{-1} + 2\pi \int_1^{-1} \Big[(1-\tilde{s}^2)f'(\tilde{s}) - 2\tilde{s}f(\tilde{s}) \Big] d\tilde{s} \tag{2.703}$$

$$+ O(R^{-2\delta} + \varepsilon) \tag{2.704}$$

$$=O(R^{-2\delta} + \varepsilon) \tag{2.705}$$

2.7 The case of equality

We prove the two rigidity cases of our main theorem 1. The first proof adresses the case $k = g$ and combines computations from [HJM19] with our hyperbolic rigidity theorem 48. The second proof adresses the case where k is merely asmpmtotic to g but we additionally assume $E = |P| = 0$. Then we show that (M, g, k) emebds isometrically in Minkowski space.

2.7.1 Proof for $k = g$

We first need the following elementary lemma in the spirit of Lemma 7.1 in [HKK20].

Lemma 46. *Suppose that $u \in C^2(M)$ satisfies*

$$\begin{cases} \nabla_{ij}u + k_{ij}|\nabla u| = 0 & \text{in } M, \\ u = v + O(|v|r^{-1}) & \text{at infinity.} \end{cases} \tag{2.706}$$

Then $|\nabla u| \neq 0$.

Proof. Due to our asymptotics we can find a point $x_0 \in M$ such that $|\nabla u(x_0)| \neq 0$. For any other point $x \in M$, let γ be a curve parameterized by arclength connecting x_0 to x. Observe that since there exists C such that

$$|\nabla|\nabla u|| \leq |\nabla^2 u| = |k| \cdot |\nabla u| \leq C|\nabla u|, \tag{2.707}$$

we have that

$$\left|(\log|\nabla u| \circ \gamma)'\right| \leq |\nabla \log|\nabla u|| \circ \gamma \leq C. \tag{2.708}$$

By integrating along γ, it follows that there is a constant $C_1 > 0$ depending on the distance between x and x_0 such that

$$C_1^{-1}|\nabla u(x_0)| \leq |\nabla u(x)| \leq C_1|\nabla u(x_0)|, \qquad x \in M_{r_0}. \tag{2.709}$$

The desired result follows since $|\nabla u(x_0)| \neq 0$. □

Theorem 47. *Let (M,g) be an three dimensional asymptotically hyperbolic manifold satisfying $R_g \geq -6$. If $E = \sqrt{P_1^2 + P_2^2 + P_3^3}$, then M is isometric to hyperbolic space.*

Proof. Let $E = \sqrt{P_1^2 + P_2^2 + P_3^3}$. Our ingegral formula implies $\nabla^2 u = |\nabla u| g$ and $R = -6$. Moreover, we have $\nabla u = \nabla |\nabla u|$. Following the proof of Proposition 4.5 in [HJM], we compute in a geodesic normal coordinates

$$0 = \nabla^3_{ijk}u - \nabla^3_{ikj}u - R^l_{kji}\nabla_l u = \nabla_k u g_{ij} - \nabla_j u g_{ik} - R^l_{kji}\nabla_l u. \tag{2.710}$$

Since $|\nabla u| \neq 0$, we may define the unit vector $\nu = \frac{\nabla u}{|\nabla u|}$. Choose e_1, e_2 such that e_1, e_2, ν are an orthnormal frame at a point $p \in M_{ext}$. By construction we have $\nabla_1 u = \nabla_2 u = 0$. Hence we have for the sectional curvature

$$K(e_1 \wedge \nu)|\nabla u| = R^i_{1\nu 1}\nabla_i u = -\nabla_\nu g_{11} = |\nabla u| \tag{2.711}$$

where we used (2.710). Hence we have $K(e_1 \wedge \nu) = -1$ and analogously $K(e_2 \wedge \nu) = -1$. Since $R = -6$ on M_{ext}, we also have $K(e_1 \wedge e_2) = -1$. Hence M_{ext} is hyperbolic which implies M is hyperbolic outside a large ball. Moreover, we have $R \geq -6$ on M. Thus, the result follows from the following well known proposition. □

Theorem 48. *Let (M,g) be a three dimensional, non-compact, complete Riemannian manifold with one end which is exactly hyperbolic, i.e. $g = g_{\mathbb{H}}$ outside some large ball B_ρ. Moreover, suppose that $R_g \geq -6$ everywhere. Then M is isometric to hyperbolic* *space.*

Proof. We consider the initial data set $(\hat{M}, \hat{g}, k)$: near infinity, outside B_ρ the hyperbolic end of (M, g) embeds into Minkowski spacetime. In Minkowski spacetime, we smoothly interpolate between the hyperbolic space and the flat slice $(\mathbb{R}^3, \delta)$. We call this interpolation construction between (M, g) and $(\mathbb{R}^3, \delta)$, $(\hat{M}, \hat{g})$. Next, we define the tensor k to be the second fundamental form of $(\hat{M}, g) \subset \mathbb{R}^{3,1}$ near infinity, and $k = g$ in the interior. Note that this construction is smooth since in the hyperboloidal model the second fundamental form equals the metric tensor.

The constructed initial data set $(\hat{M}, \hat{g}, k)$ satisfies the dominant energy condition. Hence the spacetime PMT applies to show that $(\hat{M}, \hat{g}, k)$ is isometric to a subset of Minkowski space. Consider the function $C : B_\rho \subset M \to \mathbb{R}^{3,1}$ given by

$$C(x) = x - \nu(x) \tag{2.712}$$

where ν the unit normal of $M \subset \mathbb{R}^{3,1}$. We then have

$$\partial_i C(x) = e_i - k_{ij} e^j = 0 \tag{2.713}$$

for $i = 1, 2, 3$, where $\{e_i\}$ is an orthonormal basis at some point in M. Hence $C(x)$ is constant. By the definition of C this implies that $M \cap B_\rho$ is a subset of the unit sphere in Minkowski space, i.e. $M = \mathbb{H}^3$. □

2.7.2 Proof for $E = |P| = 0$

The rigidity proof in [HKK20] can be adapted in the hyperbolic setting. However, we use a different approach to prove the rigidity case based on the observation that the level set is flat in the case equality.

Lemma 49. *If $E = |P|$, then the level set of u is flat.*

Proof. Since in the case equality,

$$\nabla^2 u + |\nabla u| k = 0, \tag{2.714}$$

$$\mu |\nabla u| + \langle J, \nabla u \rangle = 0, \tag{2.715}$$

when $|\nabla u| \neq 0$, we use Formula (3.21) in [HKK20],

$$\begin{aligned}
\Delta |\nabla u| =& \frac{1}{2|\nabla u|}(|\nabla^2 u + |\nabla u| k|^2 + (2\mu - R_\Sigma)|\nabla u|^2 - 2\langle k, \nabla^2 u\rangle |\nabla u| & (2.716)\\
&- 2\langle \nabla u, \nabla \mathrm{tr}(k)\rangle |\nabla u|) & (2.717)\\
=& \frac{1}{2|\nabla u|}((2\mu - R_\Sigma)|\nabla u|^2 + 2|k|^2|\nabla u|^2 + 2\langle J - \operatorname{div} k, \nabla u\rangle |\nabla u|) & (2.718)\\
=& \frac{1}{2|\nabla u|}(-R_\Sigma |\nabla u|^2 + 2|k|^2|\nabla u|^2 - 2\langle \mathrm{div} k, \nabla u\rangle |\nabla u|). & (2.719)
\end{aligned}$$

Let us apply the Hessian equation to recompute $\Delta|\nabla u|$

$$\begin{aligned}
\Delta |\nabla u| =& \nabla^i \nabla_i (\nabla_j u \nabla^j u)^{\frac{1}{2}} & (2.720)\\
=& \nabla^i [(\nabla_i \nabla_j u)(\nabla^j u)|\nabla u|^{-1}] & (2.721)\\
=& -\nabla^i (k_{ij} \nabla^j u) & (2.722)\\
=& |k|^2 |\nabla u| - \langle \operatorname{div} k, \nabla u\rangle. & (2.723)
\end{aligned}$$

Combining Line (2.719) and (2.723), we have $R_\Sigma = 0$.

Since $u \approx -t - x$, then the level set of u is asymptotically flat. Therefore, the level set is flat. □

Second, we construct the spacetime metric as below.

Lemma 50. *Suppose u satisfies the spacetime Hessian equation $\nabla^2 u + |\nabla u| k = 0$, and $|\nabla u| \neq 0$. Let $(\mathbb{R} \times M, \hat{g})$ be a Lorentzian manifold with the metric $\bar{g}$,*

$$\hat{g} := d\tau \otimes du + du \otimes d\tau + g, \tag{2.724}$$

then we can embed M into $(\mathbb{R} \times M, \mathbf{g})$ with the given second fundamental form k.

Moreover, the null vector ∂_τ is convariantly constant, i.e., $\hat{\nabla}\partial_\tau = 0$, where $\hat{\nabla}$ is the convariant deriative with respect to $\hat{g}$.

Proof. We rewrite $\hat{g} = d\tau \otimes du + du \otimes d\tau + |\nabla u|^{-2} du^2 + g_{\Sigma_u}$, then

$$\hat{g}^{-1} = \begin{pmatrix} -|\nabla u|^2 & 1 & \\ 1 & 0 & \\ & & g_{\Sigma_u}^{-1} \end{pmatrix}. \tag{2.725}$$

We need to compute the second fundamental form of the embedding Π_{ij},

$$\begin{aligned} \Pi_{ij} =& - \langle \hat{\nabla}_{\partial_i} \partial_j, N \rangle && (2.726)\\ =& - \langle \hat{\Gamma}^\tau_{ij} \partial_t, N \rangle && (2.727)\\ =& |\nabla u| \hat{\Gamma}^\tau_{ij} && (2.728)\\ =& \frac{1}{2} |\nabla u| \hat{g}^{\tau l} (\hat{g}_{il,j} + \hat{g}_{jl,i} - \hat{g}_{ij,l}). && (2.729) \end{aligned}$$

Then

$$\Pi_{uu} = \frac{1}{2}|\nabla u|\hat{g}_{uu,u} \tag{2.730}$$
$$= \frac{1}{2}|\nabla u|\partial_u|\nabla u|^{-2} \tag{2.731}$$
$$= -|\nabla u|^{-2}\partial_u|\nabla u| \tag{2.732}$$
$$= k(\partial_u, \partial_u). \tag{2.733}$$

Suppose ∂_α, ∂_β are the tangent vectors on Σ_u,

$$\Pi_{u\alpha} = \frac{1}{2}|\nabla u|g_{uu,\alpha} = \frac{1}{2}|\nabla u|\partial_\alpha|\nabla u|^{-2} = k(\partial_u, \partial_\alpha). \tag{2.734}$$

Let $A_{\alpha\beta}$ be the second fundamental form of the level set Σ_u in M, then $A = \frac{\nabla^2 u|_{\Sigma_u}}{|\nabla u|}$. Therefore,

$$\Pi_{\alpha\beta} = -\frac{1}{2}|\nabla u|g_{\alpha\beta,u} = -\frac{|\nabla u|}{2}\frac{\partial g_{\Sigma_u}}{\partial u} = -A_{\alpha\beta} = k_{\alpha\beta}. \tag{2.735}$$

Since for any i, j, $\hat{g}_{\tau i,j} = \hat{g}_{ij,\tau} = 0$, then $\hat{\nabla}\partial_\tau = 0$. □

Before we prove the main rigidity theorem, we need to prove a technical lemma.

Lemma 51. *Suppose u^x, u^y, u^z are the spacetime harmonic functions with different asymptotics, i.e., $u^x \to -t - x$ etc.*

If $E = |P| = 0$, then at each point, $\dim(\mathrm{span}\{\nabla u^x, \nabla u^y, \nabla u^z\}) \geq 2$.

Proof. Let $L = c_x\nabla u^x + c_y\nabla u^y + c_z\nabla u^z$, where c_x, c_y and c_z are constant, not all vanishing. We define an auxiliary function $l = c_x|\nabla u^x| + c_y|\nabla u^y| + c_z|\nabla u^z|$.

Since $E = |P| = 0$, then $\nabla^2 u^x = -|\nabla u^x|k$, $\nabla^2 u^y = -|\nabla u^y|k$, and $\nabla^2 u^z = -|\nabla u^z|k$. Therefore,

$$\nabla_i L_j = -lk_{ij}, \quad \partial_i l = -k_{ij}L^j. \tag{2.736}$$

Therefore, $\partial_i(l^2 - |L|^2) = 0$, then $l^2 - |L|^2 = constant$.

Since $|\nabla u^x| \to t + x$, and similar for u^y, u^z, then we have

$$l \to c_x(t + x) + c_y(t + y) + c_z(t + z). \tag{2.737}$$

Since in hyperbolic space $\nabla(t + x) \cdot \nabla(t + y) = (t + x)(t + y) - 1$, then at infinity,

$$|L|^2 \to [c_x(t + x) + c_y(t + y) + c_z(t + z)]^2 - 2(c_x c_y + c_y c_z + c_z c_x). \tag{2.738}$$

Therefore,

$$l^2 - |L|^2 \to 2(c_x c_y + c_y c_z + c_z c_x). \tag{2.739}$$

Then $l^2 - |L|^2 = 2(c_x c_y + c_y c_z + c_z c_x)$. If $c_x c_y + c_y c_z + c_z c_x < 0$, then $L \neq 0$.

Suppose at p, the dimension of the subspace spanned by $\{\nabla u^x, \nabla u^y, \nabla u^z\}$ is one. Since $|\nabla u^x| \neq 0$, then we can assume $\nabla u^x = c_y \nabla u^y = c_z \nabla u^z$ at p. Therefore, $L_1 := \nabla u^x - c_y \nabla u^y$, $L_2 := \nabla u^x - c_z \nabla u^z$, and $L_3 := c_y \nabla u^y - c_z \nabla u^z$ vanish at p.

Then $c_y < 0$, $c_z < 0$, $c_y c_z < 0$, we have a contradiction. □

Finally, we can prove the main rigidity theorem in this section.

Theorem 52. *Let (M, g, k) be a three dimensional, simply connected, complete, asymptotically hyperbolic initial data set for the Einstein equations satisfying the dominant energy condition. If $E = |P| = 0$ in one of the asymptotic ends, then (M, g, k) arises from an isometric embedding into Minkowski space.*

Proof. We want to show that the manifold $\bar{M} := (\mathbb{R} \times M, \hat{g})$ is the Minkowski space. To see that, we verify the vanishing of the sectional curvatures.

Let N be the normal vector of M, $\nu = \frac{\nabla u}{|\nabla u|}$, then

$$N = \frac{\partial_\tau - \nabla u}{|\nabla u|}. \tag{2.740}$$

We denote $\{e_1, e_2\}$ as the orthonormal vectors on u's level set.

$E = |P| = 0$ implies $\mu = |J| = 0$, then

$$G(N, N) = G(N, \nu) = G(N, e_i) = 0, \ i = 1, 2. \tag{2.741}$$

where G is the Einstein tensor in $\bar{M}$.

We have a covariantly constant null vector $\partial_\tau = |\nabla u|(\nu + N)$ in the ambient space $\mathbb{R} \times M$. Then $\hat{\text{Ric}}(\nu, \nu) = \hat{\text{Ric}}(-N, \nu) = G(-N, \nu) = 0$. Therefore,

$$0 = \hat{\text{Ric}}(\nu, \nu) \tag{2.742}$$

$$= \hat{R}(\nu, N, N, \nu) + \hat{R}(\nu, e_1, e_1, \nu) + \hat{R}(\nu, e_2, e_2, \nu) \tag{2.743}$$

$$= \hat{R}(\nu, e_1, e_1, \nu) + \hat{R}(\nu, e_2, e_2, \nu), \tag{2.744}$$

where $\hat{R}(\nu, N, N, \nu) = 0$, as ∂_τ is convariantly constant.

We can also have $\hat{R}(e_1, e_2, \nu, e_1) = \hat{R}(e_2, e_1, \nu, e_2) = 0$, since

$$0 = G(-N, e_1) = \hat{\text{Ric}}(\nu, e_1) = \hat{R}(\nu, e_2, e_2, e_1). \tag{2.745}$$

We want to show the Riemannian curvatures of $\bar{M}$ vanish. Let A_{ij} be the second fundemental form of Σ_u in M, then $A_{ij} = -k_{ij}|_\Sigma$. Let $\hat{R}$ be the Riemmanian curvature in $(\mathbb{R} \times M, \hat{g})$, then

$$\hat{R}(e_1, e_2, e_2, e_1) = R^{\Sigma}(e_1, e_2, e_2, e_1) - k_{12}k_{21} + k_{22}k_{11} + A_{12}A_{21} - A_{22}A_{11} = 0. \tag{2.746}$$

Note that the above computation of $\hat{R}$ only relies on the flatness of the level set of u and $A_{ij} = -k_{ij}|_{\Sigma_u}$, although the definition of $\hat{g}$ depends on u. Then we can choose another spacetime harmonic function $\tilde{u}$ with a different asymptotic. Similarly, we have the level set of $\tilde{u}$ is flat and the second fundamental form of the level set is $-k_{ij}|_{\Sigma_{\tilde{u}}}$. Then we have $\hat{R}(\tilde{e}_1, \tilde{e}_2, \tilde{e}_2, \tilde{e}_1) = 0$, where $\{\tilde{e}_1, \tilde{e}_2\}$ are the orthonormal basis of $\Sigma_{\tilde{u}}$. According to the Lemma 51, at a given point p, we can pick $\tilde{u}$ such that $\{\nabla\tilde{u}, \nabla u\}$ are linearly independent.

Suppose at p, let $e_1 = \tilde{e}_1 \in T\Sigma_u \cap T\Sigma_{\tilde{u}}$. Then we have $\hat{R}(e_1, e_2, e_2, e_1) = \hat{R}(e_1, \hat{e}_2, \hat{e}_2, e_1) = 0$, and $e_2 \neq \hat{e}_2$. As $\hat{R}(e_1, e_2, \nu, e_1) = 0$, then $\hat{R}(e_1, \nu, \nu, e_1) = 0$. From Line (2.744), then we have $\hat{R}(e_2, \nu, \nu, e_2) = 0$. Therefore, $\hat{R} = 0$, since we have the covariantly constant null vector. Then $\bar{M}$ is the Minkowski spacetime. □

In this proof, we use three harmonic functions as [HKK20]. If $E = |P|$, we only have one spacetime harmonic function satisfying the spacetime Hessian equation. It is not clear whether we can reduce the number of harmonic functions in the proof.

Appendix 2.A Boost invariance

Let τ be a boost on Minkowski spacetime, then we can induce a coordinate transformation on M. Suppose $(\tilde{P}, \tilde{E})$ be the energy momentum vector after the boost, then (see [Mic11], [CH03], [Sak20]), we have $(\tilde{P}, \tilde{E}) = \tau(P, E)$.

Let u_a be the solution of $\Delta u + \mathcal{K}|\nabla u| = 0$ and u_a is asymptotic to $-t - a \cdot x$ with $|a| = 1$, when $r \to \infty$. Suppose in the boosted coordinate, $\tilde{u}_{\tilde{a}}$ is the solution of $\Delta u + \mathcal{K}|\nabla u| = 0$ and asymptotic to $-\tilde{t} - \tilde{a} \cdot \tilde{x}$, where $\tau(x, t) = (\tilde{x}, \tilde{t})$ and $\tau^{-1}(-a, 1) = c(-\tilde{a}, 1)$, $c \in \mathbb{R}^+$.

Since

$$-\tilde{t} - \tilde{a} \cdot \tilde{x} = \langle (-\tilde{a}, 1), (\tilde{x}, \tilde{t}) \rangle = c^{-1} \langle \tau^{-1}(-a, 1), \tau(x, t) \rangle = c^{-1} \langle (-a, 1), (x, t) \rangle = c^{-1}(-t - a \cdot x),$$

where $\langle \, , \rangle$ is the Minkowski metric. Therefore, $\tilde{u}_{\tilde{a}} = c^{-1} u_a$.

Since $(\tilde{P}, \tilde{E}) = \tau(P, E)$, similar to the computation above, we have $\tilde{E} + \tilde{a} \cdot \tilde{P} = c^{-1}(E + a \cdot P)$. Therefore, after a boost, our main inequality remains the same up to a scaling.

Appendix 2.B Linear expansion

The metric in this section is purely hyperbolic space. For convenience, we use Δ and ∇ as the Laplacian and gradient in the hyperbolic space.

Here is a technique lemma that we use in the following computation.

Lemma 53. *Here are some formulas for $|v|$ and $t = \sqrt{1+r^2}$.*

1. $|\nabla v| = |v|$.
2. $\Delta |v| = 3|v|$, $\Delta t = 3t$.
3. $\nabla^2 |v| = |v| b$, *and* $\nabla^2 t = tb$, *where b is the hyperbolic metric*
4. $\nabla |v| \cdot \nabla t = |v|t - 1$
5. $|\nabla t|^2 = t^2 - 1$.

We have a barrier for w, $w = u - v = O(|v|^a t^{-a})$, where $a = \min\{\frac{3}{2}, \frac{1+\tau}{2}\}$. We denote $u = v + |v|^a t^{-a}\psi$. Since $\nabla t \cdot \nabla |v| = t|v| - 1$, we have

$$\Delta(|v|^a t^{-a}) = t^{-a}\Delta|v|^a + |v|^a \Delta t^{-a} + 2\nabla |v|^a \cdot \nabla t^{-a} \tag{2.747}$$

$$= a(a+2)|v|^a t^{-a} + |v|^a \left[a(a-2)t^{-a} - a(a+1)t^{-a-2}\right] \tag{2.748}$$

$$- 2a^2 |v|^{a-1} t^{-a-1}(t|v| - 1) \tag{2.749}$$

$$= 2a^2 |v|^{a-1} t^{-a-1} - a(a+1)|v|^a t^{-a-2}, \tag{2.750}$$

then

$$\Delta(v + |v|^a t^{-a}\psi) \tag{2.751}$$

$$= 3v + |v|^a t^{-a}\Delta\psi + \left[2a^2|v|^{a-1}t^{-a-1} - a(a+1)|v|^a t^{-a-2}\right]\psi \tag{2.752}$$

$$+ 2\nabla(|v|^a t^{-a}) \cdot \nabla\psi. \tag{2.753}$$

To expand $|\nabla u|$, let us compute $|\nabla u|^2$. First, we have

$$|\nabla(|v|^a t^{-a})|^2 = \left|a|v|^{a-1}t^{-a}\nabla|v| - a|v|^a t^{-a-1}\nabla t\right|^2 \tag{2.754}$$

$$=a^2\left[|v|^{2a}t^{-2a} - 2|v|^{2a}t^{-2a} + 2|v|^{2a-1}t^{-2a-1} + |v|^{2a}t^{-2a} - |v|^{2a}t^{-2a-2}\right] \tag{2.755}$$

$$=2a^2|v|^{2a-1}t^{-2a-1} - a^2|v|^{2a}t^{-2a-2}, \tag{2.756}$$

then we can expand $|\nabla u|^2$ as below

$$|\nabla u|^2 \tag{2.757}$$

$$=\left|-\nabla|v| + \nabla(|v|^a t^{-a}\psi)\right|^2 \tag{2.758}$$

$$=|v|^2 - 2\nabla|v|\cdot\nabla(|v|^a t^{-a}\psi) + \left|\nabla(|v|^a t^{-a}\psi)\right|^2 \tag{2.759}$$

$$=|v|^2 - 2a|v|^a t^{-a-1}\psi - 2|v|^a t^{-a}\nabla|v|\cdot\nabla\psi + |v|^{2a}t^{-2a}|\nabla\psi|^2 \tag{2.760}$$

$$+ 2|v|^a t^{-a}\psi\nabla(|v|^a t^{-a})\cdot\nabla\psi + \left(2a^2|v|^{2a-1}t^{-2a-1} - a^2|v|^{2a}t^{-2a-2}\right)\psi^2. \tag{2.761}$$

According to the Taylor expansion $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{1}{16}x^3 + O(x^4)$, and $\psi = O_2(1)$, we have

$$|\nabla u| =|v|\Bigg[1 - a|v|^{a-2}t^{-a-1}\psi - |v|^{a-1}t^{-a}\frac{\nabla|v|}{|v|}\cdot\nabla\psi + \frac{1}{2}|v|^{2a-2}t^{-2a}|\nabla\psi|^2 \tag{2.762}$$

$$+ |v|^{a-2}t^{-a}\psi\nabla(|v|^a t^{-a})\cdot\nabla\psi + \left(a^2|v|^{2a-3}t^{-2a-1} - \frac{a^2}{2}|v|^{2a-2}t^{-2a-2}\right)\psi^2 \tag{2.763}$$

$$- \frac{1}{2}|v|^{2a-2}t^{-2a}\left|\frac{\nabla|v|}{|v|}\cdot\nabla\psi\right|^2 - a|v|^{2a-3}t^{-2a-1}\psi\frac{\nabla|v|}{|v|}\cdot\nabla\psi \tag{2.764}$$

$$- \frac{a^2}{2}|v|^{2a-4}t^{-2a-2}\psi^2 + \frac{1}{2}|v|^{3a-3}t^{-3a}|\nabla\psi|^2\frac{\nabla|v|}{|v|}\cdot\nabla\psi \tag{2.765}$$

$$+ \frac{1}{2}|v|^{3a-3}t^{-3a}\left(\frac{\nabla|v|}{|v|}\cdot\nabla\psi\right)^3 + O_2(|v|^{3a-\frac{7}{2}}t^{-3a-\frac{1}{2}})\Bigg]. \tag{2.766}$$

Therefore,

$$
\begin{aligned}
&|v|^{-a}t^{a}(\Delta u+3|\nabla u|) && (2.767)\\
=&\Delta\psi+\left(2a^2|v|^{-1}t^{-1}-a(a+1)t^{-2}\right)\psi+2a\left(\frac{\nabla|v|}{|v|}-\frac{\nabla t}{t}\right)\cdot\nabla\psi && (2.768)\\
&-3a|v|^{-1}t^{-1}\psi-3\frac{\nabla|v|}{|v|}\cdot\nabla\psi+\frac{3}{2}|v|^{a-1}t^{-a}|\nabla\psi|^2 && (2.769)\\
&+3|v|^{-1}\psi\nabla(|v|^{\frac{3}{2}}t^{-\frac{3}{2}})\cdot\nabla\psi+\left(3a^2|v|^{a-2}t^{-a-1}-\frac{3a^2}{2}|v|^{a-1}t^{-a-2}\right)\psi^2 && (2.770)\\
&-\frac{3}{2}|v|^{a-1}t^{-a}\left|\frac{\nabla|v|}{|v|}\cdot\nabla\psi\right|^2-3a|v|^{a-2}t^{-a-1}\psi\frac{\nabla|v|}{|v|}\cdot\nabla\psi-\frac{3a^2}{2}|v|^{a-3}t^{-a-2}\psi^2 && (2.771)\\
&+\frac{3}{2}|v|^{2a-2}t^{-2a}|\nabla\psi|^2\frac{\nabla|v|}{|v|}\cdot\nabla\psi+\frac{3}{2}|v|^{2a-2}t^{-2a}\left(\frac{\nabla|v|}{|v|}\cdot\nabla\psi\right)^3+O_2(|v|^{2a-\frac{5}{2}}t^{-2a-\frac{1}{2}}) && (2.772)\\
=&\Delta\psi-\left[2a\frac{\nabla t}{t}-(2a-3)\frac{\nabla|v|}{|v|}\right]\cdot\nabla\psi-\left[(3a-2a^2)|v|^{-1}t^{-1}+a(a+1)t^{-2}\right]\psi && (2.773)\\
&+\left(3a^2|v|^{a-2}t^{-a-1}-\frac{3a^2}{2}|v|^{a-1}t^{-a-2}-\frac{3a^2}{2}|v|^{a-3}t^{-a-2}\right)\psi^2 && (2.774)\\
&+\frac{3}{2}|v|^{a-1}t^{-a}\left(|\nabla\psi|^2-\left|\frac{\nabla|v|}{|v|}\cdot\nabla\psi\right|^2\right)+3|v|^{-1}\psi\nabla(|v|^{\frac{3}{2}}t^{-\frac{3}{2}})\cdot\nabla\psi && (2.775)\\
&-3a|v|^{a-2}t^{-a-1}\psi\frac{\nabla|v|}{|v|}\cdot\nabla\psi+\frac{3}{2}|v|^{2a-2}t^{-2a}|\nabla\psi|^2\frac{\nabla|v|}{|v|}\cdot\nabla\psi && (2.776)\\
&+\frac{3}{2}|v|^{2a-2}t^{-2a}\left(\frac{\nabla|v|}{|v|}\cdot\nabla\psi\right)^3+O_2(|v|^{2a-\frac{5}{2}}t^{-2a-\frac{1}{2}}). && (2.777)
\end{aligned}
$$

Chapter 3

Scalar curvature volume comparison theorem

Bishop theorem is a classical theorem in differential geometry that establishes the connection between volume and Ricci curvature. It was proven by Bishop in 1963 ([Bis63]).

We assume throughout this chapter that (M, g) is a compact smooth n-dimensional Riemannian manifold. Let $(S^n, \bar{g})$ be the unit n-sphere with standard metric, i.e. it has constant sectional curvature 1. Let Ric_g, R_g and $\mathrm{vol}(M)$ be the Ricci curvature tensor, scalar curvature and volume of (M, g), respectively.

Theorem 54 (Bishop theorem)**.** *If* $\mathrm{Ric}_g \geq (n-1)g$*, then* $\mathrm{vol}(M) \leq \mathrm{vol}(S^n)$*.*

Aside from the classical proof [Pet16], there is an optimal transport approach in Lott, Villani and Strum's seminal papers (see [LV09],[Stu06a],[Stu06b]). They defined a synthetic Ricci curvature on metric measure spaces using optimal transport. Thus Bishop theorem can be generalized to metric measure spaces. The third approach was discovered by H.Bray in his thesis using isoperimetric surfaces ([Bra09]). A byproduct of the third approach is Bray's football theorem which is a volume comparison theorem involving scalar curvature in dimension 3.

Theorem 55 (Bray's football theorem)**.** *Let* (M, g) *be a three dimensional Riemannian manifold satisfying* $\mathrm{Ric}_g \geq \varepsilon(n-1)g$*,* $R_g \geq n(n-1)$*,* $\varepsilon \in (0, 1)$*, then* $V_g \leq \alpha(\varepsilon)V_{\bar{g}}$*, where* $\alpha(\varepsilon) = 1$*, when* $\varepsilon \in [\varepsilon_0, 1)$*;* $\alpha(\varepsilon) > 1$*, when* $\varepsilon \in (0, \varepsilon_0)$*.*

For a full expression of $\alpha(\varepsilon)$, the readers can find it in ([Bra09],[GV04] and [BGLZ19]). Regarding the constant ε_0 in Theorem 1.2, numerical tests suggest $0.134 < \varepsilon_0 < 0.135$; M. Gurskya and J. Viaclovskyb proved $\varepsilon_0 \leq 0.5$. When $\varepsilon \in (0, \varepsilon_0)$, (M, g) with the largest volume in Theorem 1.2 is axisymmetric, i.e. (M, g) has the shape of a football (American football). For the case of axisymmetric in higher dimensions, we have the following theorem.

Theorem 56. *Let $n \geq 3$. Let (M, g) be an n-dimensional, axisymmetric Riemannian manifold , i.e., $M = [0, a] \times_f S^{n-1}$, $g = dt^2 + f(t)^2 d\sigma^2$, where $d\sigma^2$ is the standard metric of S^{n-1}. There exists an $\varepsilon(n) < 1$, such that for any axisymmetric manifold (M, g) satisfies:*

$$\mathrm{Ric}(g) \geq \varepsilon(n) \cdot \mathrm{Ric}_0 \cdot g \text{ and } R(g) \geq R_0,$$

we have: $\mathrm{vol}(M) \leq \mathrm{vol}(S^n)$.

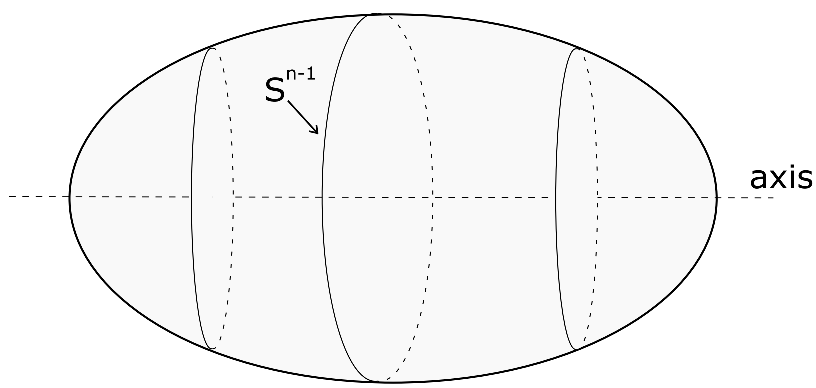

Figure 3.1: Axisymmetric Manifold

If the manifold has a uniform upper bound for Ricci curvature, we have Theorem 57 which is a high dimensional analog of Bray's football theorem.

Theorem 57. *For any $C > 0$, there exists an $\varepsilon = \varepsilon(n, C) \in (0, 1)$, such that for any compact Riemannian manifold (M, g) satisfies*

1. $(1-\varepsilon)(n-1)g \leq \mathrm{Ric}_g \leq Cg$,

2. $R_g \geq n(n-1)$,

then $\mathrm{vol}(M) \leq \mathrm{vol}(S^n)$.

If we choose ε sufficiently small, then the results in [CC97] show that M is diffeomorphic to S^n. According to Andersen's paper [And90], g is close to $\bar{g}$ in $C^{1,\alpha}$ norm. Hence, Theorem 57 is a directly result of our main Theorem 60. We use the tools in [BM11] to prove the main Theorem 60. The perturbation formula of the scalar curvature is crucial for deriving the contradiction. Theorem 57 is slightly stronger than Corollary A in [Yua16], while Corollary A needs the metric g on S^n is close to the standard metric in $W^{2,p}$ norm.

3.1 Proof of Theorem 56

In the section, we only focus on $n \geq 4$, since $n = 3$ is included in Bray's football theorem.

As $g = dt^2 + f(t)^2 d\sigma^2$, $t \in (0, a)$, according to O'Neil ([O'n83]), for vertical tangent vectors $v, w \in TS^{n-1}$,

$$\mathrm{Ric}_g(v, w) = \mathrm{Ric}(v, w)^{S^{n-1}} - \langle v, w \rangle f^{\#}, \tag{3.1}$$

$$\text{where } f^{\#} = \frac{\Delta f}{f} + (n-2)\frac{\langle \nabla f, \nabla f \rangle}{f^2}, \tag{3.2}$$

$$\mathrm{Ric}_g(\partial_t, \partial_t) = -\frac{n-1}{f} f''. \tag{3.3}$$

Since $0, a$ are two end points, we have $f(0) = f(a) = 0$ and $f(t) > 0$, when $t \in (0, a)$. Since we assume g to be smooth, then f is smooth.

Without losing of generality, we may assume $f(t) \geq 0$, $f'(0) \geq 0$, then the curvature conditions in Theorem 1.3 imply:

$$-\frac{f''}{f} \geq \varepsilon, \tag{3.4}$$

$$\frac{n-2}{f^2} - \frac{f''}{f} - (n-2)\frac{(f')^2}{f^2} \geq (n-1)\varepsilon, \tag{3.5}$$

$$-\frac{2f''}{f} + \frac{n-2}{f^2} - (n-2)\frac{(f')^2}{f^2} \geq n, \tag{3.6}$$

where $\varepsilon \in (0, 1]$.

Denote $C_\varepsilon(f) = \mathrm{vol}(M)/\mathrm{vol}(S^n)$, then,

$$C_\varepsilon(f) = \frac{\omega_{n-1} \int_0^a f(t)^{n-1} dt}{\omega_n},$$

where ω_n is the volume of S^n.

Note that when $f = \sin t$, (M, g) is an n-sphere with standard metric and f satisfies equations (3.4)-(3.6), $C_\varepsilon(\sin t) = 1$.

Hence, we need to prove: if ε is sightly less that 1, we still have $C_\varepsilon(f) \leq 1$.

Before proving Theorem 1.3, we need the following lemma.

Lemma 58. *For $x \in [\frac{1}{2}, \infty)$,*

$$\sqrt{x - \frac{1}{2}} \cdot \frac{\Gamma(x)}{\Gamma(x + \frac{1}{2})} < \sqrt{x + \frac{1}{2}} \cdot \frac{\Gamma(x+1)}{\Gamma(x + \frac{3}{2})}.$$

Proof. Since $\Gamma(x+1) = x\Gamma(x)$, therefore,

$$\sqrt{x - \frac{1}{2}} \cdot \frac{\Gamma(x)}{\Gamma(x + \frac{1}{2})} = \sqrt{x - \frac{1}{2}} \cdot \frac{\Gamma(x+1)}{\Gamma(x + \frac{3}{2})} \cdot \frac{x + \frac{1}{2}}{x} \tag{3.7}$$

$$\leq \sqrt{x + \frac{1}{2}} \cdot \frac{\Gamma(x+1)}{\Gamma(x + \frac{3}{2})}. \tag{3.8}$$

□

In fact, $\sqrt{x - \frac{1}{2}} \cdot \frac{\Gamma(x)}{\Gamma(x+\frac{1}{2})}$ is increasing on $[\frac{1}{2}, \infty)$, however, lemma 58 is enough for us to prove Theorem 1.3.

Line (3.4) implies $f(t)$ is concave, we can assume $f'(t) \geq 0$, for $t \in [0, r]$, where r satisfies $f'(r) = 0$. We only focus on $[0, r]$, since the case of $[r, a]$ is similar to $[0, r]$ by symmetry.

From Line (3.6), we have

$$\frac{d}{dt} f^{n-2}(1 - (f')^2 - f^2) \tag{3.9}$$

$$= (n-2) f^{n-3} f' - (n-2) f^{n-3} (f')^3 - 2 f^{n-2} f'' - n f^{n-1} f' \tag{3.10}$$

$$= f^{n-1} f' \left[-\frac{2f''}{f} + \frac{n-2}{f^2} - (n-2)\frac{(f')^2}{f^2} - n \right] \geq 0. \tag{3.11}$$

Hence, $f^2 + f'^2 \leq 1$. Suppose we only have Line (3.4) and Line (3.6), then

$$\frac{n-2}{f^2} - \frac{f''}{f} - (n-2)\frac{(f')^2}{f^2} \geq n-2-\frac{f''}{f} \geq n-2+\varepsilon \geq (n-1)\varepsilon. \tag{3.12}$$

Therefore, Line (3.4) and (3.6) imply (3.5).

Assume $m = f(r)$, i.e., $m = \max_{t\in[0,a]} f(t)$, then for $0 < t \leq r$, $f^{n-2}(1-(f')^2 - f^2) \leq m^{n-2}(1-m^2)$, and hence $f' \geq (1 - f^2 - \frac{m^{n-2}(1-m^2)}{f^{n-2}})^{1/2}$.

Line (3.4) implies: $\varepsilon f^2 + (f')^2$ is decreasing, so $f' \geq (\varepsilon(m^2 - f^2))^{1/2}$.

Therefore, we have two lower bounds for f', let

$$M_\varepsilon(f) := \max\{1 - f^2 - \frac{m^{n-2}(1-m^2)}{f^{n-2}}, \varepsilon(m^2 - f^2)\}. \tag{3.13}$$

Then we have

$$\int_0^r f(t)^{n-1}dt = \int_0^m \frac{f^{n-1}}{f'}df \leq \int_0^m \frac{f^{n-1}}{M_\varepsilon(f)^{1/2}}df. \tag{3.14}$$

We substitute f by ms, then for $s \in [0,1)$,

$$\begin{aligned} 1 - f^2 - \frac{m^{n-2}(1-m^2)}{f^{n-2}} &= 1 - m^2s^2 - \frac{m^{n-2}(1-m^2)}{m^{n-2}s^{n-2}} \\ &= m^2(1-s^2)\left[1 - \frac{(1-m^2)(1-s^{n-2})}{m^2s^{n-2}(1-s^2)}\right], \end{aligned}$$

and hence

$$\int_0^m \frac{f^{n-1}}{M_\varepsilon(f)^{1/2}}df = \int_0^1 \frac{m^{n-1}s^{n-1}ds}{\sqrt{\max\{(1-s^2)[1-\frac{(1-m^2)(1-s^{n-2})}{m^2s^{n-2}(1-s^2)}], \varepsilon(1-s^2)\}}}. \tag{3.15}$$

If $m^{n-1} \leq \varepsilon^{1/2}$, then

$$\int_0^m \frac{f^{n-1}}{f'} df \leq \int_0^m \frac{f^{n-1}}{M_\varepsilon(f)^{1/2}} df \leq \int_0^1 \frac{m^{n-1}t^{n-1}}{[\varepsilon(1-t^2)]^{1/2}} dt \leq \int_0^1 \frac{t^{n-1}}{(1-t^2)^{1/2}} dt, \quad (3.16)$$

i.e. $\mathrm{vol}(\{(t,\omega) \in M | t \in [0,r], \omega \in S^{n-1}\}) \leq \frac{1}{2}\mathrm{vol}(S^n)$. When $t \in [r,a]$, we have a similar argument by symmetry, $\mathrm{vol}(\{(t,\omega) \in M | t \in [r,a], \omega \in S^{n-1}\}) \leq \frac{1}{2}\mathrm{vol}(S^n)$. Therefore, when $m^{n-1} \leq \varepsilon^{1/2}$, we have proven the volume of M is not great than the volume of the standard unit S^n. Then we can focus on $m^{n-1} \geq \varepsilon^{1/2}$.

Since when $s \to 1^-$, we have $1 - \frac{(1-m^2)(1-s^{n-2})}{m^2 s^{n-2}(1-s^2)} \to 1 - \frac{(n-2)(1-m^2)}{2m^2}$, then

$$1 - \frac{(n-2)(1-m^2)}{2m^2} - \varepsilon \geq 1 - \frac{(n-2)(1-m^2)}{2m^2} - m^{2(n-1)} = (1-m^2)\left(\sum_{i=0}^{n-2} m^{2i} - \frac{n-2}{2m^2}\right).$$

If $\varepsilon \to 1$, then $m \to 1$, we have

$$\sum_{i=0}^{n-2} m^{2i} > \frac{n-2}{2m^2}.$$

Therefore, we can find an $\varepsilon \in (0,1)$ as required in the Theorem 56 such that

$$1 - \frac{(n-2)(1-m^2)}{2m^2} - \varepsilon \geq (1-m^2)\left(\sum_{i=0}^{n-2} m^{2i} - \frac{n-2}{2m^2}\right) > 0, \quad (3.17)$$

for any $\varepsilon^{\frac{1}{2(n-1)}} \leq m \leq 1$. Then, we can fix an ε sufficiently close to 1, and we define

$$h(m) := \max\{x | x \in (0,1), x \text{ satisfies } 1 - \frac{(1-m^2)(1-x^{n-2})}{m^2 x^{n-2}(1-x^2)} = \varepsilon\}.$$

Hence, for any $s \in (h(m), 1)$,

$$1 - \frac{(1-m^2)(1-s^{n-2})}{m^2 s^{n-2}(1-s^2)} > \varepsilon.$$

Since $n \geq 4$, then $h(m) = x$ satisfies

$$x^{n-2} = \frac{1-m^2}{m^2(1-\varepsilon)} \cdot \frac{1-x^{n-2}}{1-x^2} \leq \frac{1-m^2}{m^2(1-\varepsilon)} \cdot \frac{n-2}{2}.$$

Therefore, when $m \to 1$, $h(m) \to 0$

Denote $H(m) = \int_0^{h(m)} \frac{m^{n-1}t^{n-1}}{[\varepsilon(1-t^2)]^{1/2}}dt + \int_{h(m)}^1 \frac{m^{n-1}t^{n-1}}{[(1-t^2)(1-\frac{(1-m^2)(1-t^{n-2})}{m^2t^{n-2}(1-t^2)})]^{1/2}}dt.$

Then we have

$$H(m) \geq \int_0^m \frac{f^{n-1}}{M_\varepsilon(f)^{1/2}}df \geq \int_0^r f(t)^{n-1}dt.$$

Since when $m \to 1$, we have $h(m) \to 0$. If $m = 1$, then the expression of $H(1)$ is exactly the volume of hemisphere. As a result of this observation, we need to show $H(m) \leq H(1)$. To prove this, we give an estimate of $H'(m)$ as shown below.

$$\begin{aligned}
m^{4-n}H'(m) =&(n-1)\left[\int_{h(m)}^1 \frac{m^2t^{n-1}dt}{\sqrt{(1-t^2)(1-\frac{(1-m^2)(1-t^{n-2})}{m^2t^{n-2}(1-t^2)})}} + \int_0^{h(m)} \frac{m^2t^{n-1}dt}{[\varepsilon(1-t^2)]^{1/2}}\right] \\
&- \int_{h(m)}^1 \frac{t(1-t^{n-2})dt}{(1-t^2)^{3/2}(1-\frac{(1-m^2)(1-t^{n-2})}{m^2t^{n-2}(1-t^2)})^{3/2}} \\
\geq&(n-1)m^2\int_0^1 \frac{t^{n-1}dt}{(1-t^2)^{1/2}} - \varepsilon^{-3/2}\int_0^1 \frac{t}{(1-t^2)^{1/2}} \cdot \frac{1-t^{n-2}}{1-t^2}dt \\
\geq&(n-1)\varepsilon^{1/(n-1)}\int_0^1 \frac{t^{n-1}dt}{(1-t^2)^{1/2}} - \varepsilon^{-3/2}\int_0^1 \frac{t}{(1-t^2)^{1/2}} \cdot \frac{1-t^{n-2}}{1-t^2}dt.
\end{aligned}$$

In the estimate of $H'(m)$ above, we do not need to consider taking derivative of the bounds of the integral, since $h(m)$ is chosen so that the integrand of $H(m)$ is continuous at $h(m)$.

The rest part of this section is to prove the following inequality for $n \geq 4$.

$$(n-1)\int_0^1 \frac{t^{n-1}dt}{(1-t^2)^{1/2}} > \int_0^1 \frac{t}{(1-t^2)^{1/2}} \cdot \frac{1-t^{n-2}}{1-t^2}dt. \tag{3.18}$$

If inequality (3.18) holds, then we can find an $\varepsilon < 1$ such that $H'(m) > 0$, for $m \in [\varepsilon^{\frac{1}{2(n-1)}}, 1]$.

For $n \geq 4$, considering two different cases: $n = 2k+2$ and $n = 2k+1$, $k \in \mathbb{N}$, as they are slightly different.

$n = 2k+2$, $k \geq 1$:

$$(n-1)\int_0^1 \frac{t^{n-1}}{(1-t^2)^{1/2}}dt = (2k+1)\int_0^1 \frac{t^{2k+1}}{(1-t^2)^{1/2}}dt = (2k+1)\frac{\sqrt{\pi}}{2}\frac{\Gamma(k+1)}{\Gamma(k+\frac{3}{2})},$$

$$\int_0^1 \frac{t}{(1-t^2)^{1/2}} \cdot \frac{1-t^{n-2}}{1-t^2}dt = \frac{\sqrt{\pi}}{2}\sum_{j=0}^{k-1}\frac{\Gamma(j+1)}{\Gamma(j+\frac{3}{2})}.$$

According to lemma 58, to prove inequality (3.18), we need to show

$$\frac{(2k+1)}{\sqrt{k+1/2}} > \sum_{j=0}^{k-1}\frac{1}{\sqrt{j+1/2}}.$$

As $\frac{1}{\sqrt{j+1/2}} \leq \frac{2}{\sqrt{j+1}+\sqrt{j}} = 2\sqrt{j+1} - 2\sqrt{j}$, so $\displaystyle\sum_{j=0}^{k-1}\frac{1}{\sqrt{j+1/2}} \leq 2\sqrt{k} \leq \frac{(2k+1)}{\sqrt{k+1/2}}$.

$n = 2k+1$, $k \geq 2$:

$$(n-1)\int_0^1 \frac{t^{n-1}}{(1-t^2)^{1/2}}dt = 2k\int_0^1 \frac{t^{2k}}{(1-t^2)^{1/2}}dt = 2k\frac{\sqrt{\pi}}{2}\frac{\Gamma(k+1/2)}{\Gamma(k+1)},$$

$$\int_0^1 \frac{t}{(1-t^2)^{1/2}} \cdot \frac{1-t^{n-2}}{1-t^2} dt = \int_0^1 \frac{t^2}{\sqrt{1-t^2}}(1+t^2+\cdots+t^{2k-4}) + \frac{t}{(t+1)(1-t^2)^{1/2}} dt$$
$$= \frac{\sqrt{\pi}}{2} \sum_{j=1}^{k-1} \frac{\Gamma(j+1/2)}{\Gamma(j+1)} + \frac{\pi-2}{2}.$$

We need to prove: $k\sqrt{\pi}\frac{\Gamma(k+1/2)}{\Gamma(k+1)} \geq \frac{\sqrt{\pi}}{2} \sum_{j=1}^{k-1} \frac{\Gamma(j+1/2)}{\Gamma(j+1)} + \frac{\pi-2}{2}$,

dividing by $\frac{\sqrt{\pi}}{2} \cdot \sqrt{k} \cdot \frac{\Gamma(k+1/2)}{\Gamma(k+1)}$ at both sides, applying lemma 1 , all we need to show is

$$2\sqrt{k} \geq \sum_{j=1}^{k-1} \frac{1}{\sqrt{j}} + \frac{(\pi-2)/\sqrt{\pi}}{\sqrt{2} \cdot \frac{\Gamma(\frac{5}{2})}{\Gamma(3)}}. \tag{3.19}$$

As $\dfrac{(\pi-2)/\sqrt{\pi}}{\sqrt{2} \cdot \frac{\Gamma(\frac{5}{2})}{\Gamma(3)}} \approx 0.685 \leq \sqrt{2}$,

since $\dfrac{1}{\sqrt{j}} \leq \dfrac{2}{\sqrt{j+1/2}+\sqrt{j-1/2}} = 2\sqrt{j+1/2} - 2\sqrt{j-1/2}$,

then $\sqrt{2} + \sum_{j=1}^{k-1} \frac{1}{\sqrt{j}} \leq 2\sqrt{k}$. Therefore, Line (3.19) is proven.

All in all, there exists $\varepsilon < 1$ as required in Theorem 56.

3.2 Proof of Theorem 57

The following proposition is Proposition 11 in [BM11], while the original proposition is in space $W^{2,p}$. However, the proof in [BM11] can be applied to our circumstance with few modifications, since we can still split a $W^{1,p}$ symmetric two-tensor into a divergence free two-tensor and a Lie derivative of the metric.

Proposition 59. *Let $p > n$. Ω is an n-dimensional compact manifold with boundary. Let g, $\bar{g}$ be Riemannian metrics on Ω. If $\|g - \bar{g}\|_{W^{1,p}(\Omega,g)}$ is sufficiently small, there exists a diffeomorphism $\varphi : \Omega \to \Omega$, such that $\varphi|_{\partial\Omega} = id$ and $h = \varphi^*(g) - \bar{g}$ is divergence free. Moreover, there exists a positive constant C that depends on $(\Omega, \bar{g})$, such that:*

$$\|h\|_{W^{1,p}(\Omega,\bar{g})} \leq C\|g - \bar{g}\|_{W^{1,p}(\Omega,\bar{g})}.$$

Followed by Proposition 59, we may assume $g - \bar{g}$ is divergence free. Therefore, we have a volume comparison theorem for the scalar curvature.

Theorem 60. *Assume $(S^n, \bar{g})$ is the n-sphere with standard metric. Let g be another metric on S^n with the following properties:*

1. $R_g \geq R_{\bar{g}} = n(n-1)$,

2. $V_g \geq V_{\bar{g}}$,

where V_g, $V_{\bar{g}}$ is the volume of (S^n, g) and $(S^n, \bar{g})$

If $h = g - \bar{g}$ is sufficiently small in $W^{1,p}(S^n, \bar{g})$ norm, $p > \frac{n}{2}$, then $V_g = V_{\bar{g}}$, moreover, there exists a diffeomorphism $\varphi : S^n \to S^n$, such that $\varphi^*(\bar{g}) = g$.

Proof. Proposition 4 in [BM11] exhibits a pointwise estimate for R_g:

$$\begin{aligned}&|R_g - R_{\bar{g}} + \langle \mathrm{Ric}_{\bar{g}}, h\rangle - \langle \mathrm{Ric}_{\bar{g}}, h^2\rangle + \frac{1}{4}|\nabla h|^2 - \frac{1}{2}\nabla_i h_{kp}\cdot\nabla_k h_{ip}\\ &+ \frac{1}{4}|\nabla tr(h)|^2 + \nabla_i(g^{ik}g^{jl}(\nabla_k h_{jl} - \nabla_l h_{jk}))|\\ &\leq C|h|\cdot|\nabla h|^2 + C|h|^3,\end{aligned}$$

where $|\cdot|$ is the pointwise norm under $\bar{g}$, ∇ is the Levi-Civita connection of $\bar{g}$, $tr(h)$ is the trace of h under metric $\bar{g}$.

$$\begin{aligned}&\left|\int R_g - R_{\bar{g}} + (n-1)tr(h) - (n-1)|h|^2 + \frac{1}{4}|\nabla h|^2\right.\\ &\left. + \frac{1}{2}h_{kp}\cdot\nabla_i\nabla_k h_{ip} + \frac{1}{4}|\nabla tr(h)|^2 dV_{\bar{g}}\right|\\ \leq & C\int |h|\cdot|\nabla h|^2 + |h|^3 dV_{\bar{g}}.\end{aligned}$$

Since $\bar{R}_{ijkl} = \bar{g}_{il}\bar{g}_{jk} - \bar{g}_{ik}\bar{g}_{jl}$, we have:

$$\begin{aligned}\nabla_i\nabla_k h_{ip} =& \nabla_k\nabla_i h_{ip} - \bar{R}_{ikim}h_{mp} - \bar{R}_{ikpm}h_{im}\\ =& \nabla_k\nabla_i h_{ip} - tr(h)\bar{g}_{kp} + nh_{kp}.\end{aligned}$$

According to Proposition 3.1, we can assume div$h = 0$, up to a diffeomorphism

φ. Therefore,

$$\begin{aligned}
&\int (R_g - R_{\bar g})\, dV_{\bar g} \\
=&\int \Big(-(n-1)tr(h) + (n-1)|h|^2 - \frac{1}{4}|\nabla h|^2 - \frac{1}{2}h_{kp}\cdot \nabla_i\nabla_k h_{ip} - \frac{1}{4}|\nabla tr(h)|^2 \Big) dV_{\bar g} \\
&+ O(\|h\|_{C^0(S^n,\bar g)}\|h\|^2_{W^{1,2}(S^n,\bar g)}) \\
=&\int \left(-(n-1)tr(h) + (\frac{n}{2}-1)|h|^2 - \frac{1}{4}|\nabla h|^2 - \frac{1}{4}|\nabla tr(h)|^2 + \frac{1}{2}tr(h)^2 \right) dV_{\bar g} \\
&+ O(\|h\|_{C^0(S^n,\bar g)}\|h\|^2_{W^{1,2}(S^n,\bar g)}).
\end{aligned}$$

Since

$$\begin{aligned}
&V_g - V_{\bar g} \\
=&\int (\sqrt{\det(\bar g + h)} - 1) dV_{\bar g} \\
=&\int \left(\sqrt{1 + tr(h) + \frac{1}{2}tr(h)^2 - \frac{1}{2}|h|^2 + O(|h|^3)} - 1 \right) dV_{\bar g} \\
=&\int \left(\frac{1}{2}tr(h) + \frac{1}{8}tr(h)^2 - \frac{1}{4}|h|^2 \right) dV_{\bar g} + O(\|h\|_{C^0(S^n,\bar g)}\|h\|^2_{L^2(S^n,\bar g)}).
\end{aligned}$$

Denote $\delta = \frac{1}{V_{\bar g}}\int tr(h) dV_{\bar g}$, and $k = \frac{8(n-1)-4\delta}{4+\delta}$, then we have

$$\frac{k}{2} - (n-1) = -\delta(\frac{1}{2} + \frac{k}{8}).$$

Therefore,

$$\begin{aligned}
&\int (R_g - R_{\bar{g}})\, dV_{\bar{g}} + k(V_g - V_{\bar{g}}) \\
=& \int \Bigg[\left(\frac{k}{2} - n + 1\right) tr(h) + \left(\frac{k}{8} + \frac{1}{2}\right) tr(h)^2 + \left(\frac{n}{2} - 1 - \frac{k}{4}\right) |h|^2 \\
&- \frac{1}{4}|\nabla tr(h)|^2 - \frac{1}{4}|\nabla h|^2 \Bigg] dV_{\bar{g}} + O(\|h\|_{C^1(S^n,\bar{g})} \|h\|^2_{W^{1,2}(S^n,\bar{g})}) \\
=& \int \Bigg[\left(\frac{k}{8} + \frac{1}{2}\right) (tr(h) - \delta)^2 + \left(\frac{n}{2} - 1 - \frac{k}{4}\right) |h|^2 \\
&- \frac{1}{4}|\nabla tr(h)|^2 - \frac{1}{4}|\nabla h|^2 \Bigg] dV_{\bar{g}} + O(\|h\|_{C^0(S^n,\bar{g})} \|h\|^2_{W^{1,2}(S^n,\bar{g})}).
\end{aligned}$$

We apply the inequalities $|\nabla h|^2 \geq \frac{1}{n}|\nabla tr(h)|^2$, $|h|^2 \geq \frac{1}{n} tr(h)^2$, then

$$\int |h|^2 dV_{\bar{g}} \geq \frac{1}{n} \int (tr(h)^2 - \delta^2) dV_{\bar{g}} = \frac{1}{n} \int (tr(h) - \delta)^2 dV_{\bar{g}}.$$

Since $\int [tr(h) - \delta] dV_{\bar{g}} = 0$, by Poincaré inequality, we have $\|\nabla tr(h)\|^2_{L^2} \geq n\|tr(h) - \delta\|^2_{L^2}$.

Let $k = 2(n-1) - \tilde{\varepsilon}$, we have $|\tilde{\varepsilon}| \leq (n+1)|\delta|$.

Therefore, we can show:

$$
\begin{aligned}
&\int (R_g - R_{\bar{g}}) dV_{\bar{g}} + k(V_g - V_{\bar{g}}) \\
=&\int \Bigg[\left(\frac{n+1}{4} - \frac{\tilde{\varepsilon}}{8} \right) (tr(h) - \delta)^2 - (\frac{1}{2} - \frac{\tilde{\varepsilon}}{4})|h|^2 \\
&- \frac{1}{4}|\nabla tr(h)|^2 - \frac{1}{4}|\nabla h|^2 \Bigg] dV_{\bar{g}} + O(\|h\|_{C^0(S^n,\bar{g})} \|h\|^2_{W^{1,2}(S^n,\bar{g})}) \\
=&\int \Bigg[\left(\frac{n}{4}(tr(h) - \delta)^2 - \frac{1}{4}|\nabla tr(h)|^2 \right) + (\frac{1}{4} - \frac{\tilde{\varepsilon}}{8} - \frac{1}{4n})((tr(h) - \delta)^2 - |\nabla h|^2) \\
&+ \left(\frac{1}{4n}(tr(h) - \delta)^2 - \frac{1}{4}|h|^2 \right) - (\frac{1}{4} - \frac{\tilde{\varepsilon}}{4})|h|^2 - (\frac{1}{4n} + \frac{\tilde{\varepsilon}}{8})|\nabla h|^2 \Bigg] dV_{\bar{g}} \\
&+ O(\|h\|_{C^0(S^n,\bar{g})} \|h\|^2_{W^{1,2}(S^n,\bar{g})}) \\
\leq& - (\frac{1}{4n} + \frac{\tilde{\varepsilon}}{8}) \|h\|^2_{W^{1,2}(S^n,\bar{g})} + O(\|h\|_{C^0(S^n,\bar{g})} \|h\|^2_{W^{1,2}(S^n,\bar{g})}) \leq 0
\end{aligned}
$$

So we have $h = 0$, $g = \bar{g}$. □

Then combining Theorem 1.2 in [And90], we have Theorem 1.4.

Appendix 3.A $\varepsilon_0 \le 0.2$ in Bray's football theorem

This appendix is to proof the constant ε_0 in Bray's football theorem less than 0.2. The main argument is Theorem 62 at the end of this appendix.

The case of equality in Bray's football theorem can be approximate by axisymmetric manifolds. If we allow singularity at two pointy end, then we can construct an axisymmetric manifold which satisfies the curvature condition in Theorem 55 and has the maximal volume $\alpha(\varepsilon)$. Therefore, to prove $\varepsilon_0 \le 0.2$, we can focus on axisymmetric manifolds.

We use the setups in Section 3.1. Suppose (M^3, g) is a smooth axisymmetric manifold, then $g = dt^2 + f^2(t)d\sigma^2$, $t \in (0, a)$. Let r be the maximal point of $f(t)$, and $m = f(r)$. Since $f(t)$ is concave, if we want to prove $\text{vol}(M^3, g) \le \text{vol}(S^3)$, then by symmetry, we only need to show

$$\text{vol}\{(t,p) \in M^3 | t \in [0,r], p \in S^2\} \le \frac{1}{2}\text{vol}(S^3).$$

Therefore, we need to obtain

$$\int_0^r 4\pi f^2(t)dt \le \frac{1}{2}\text{vol}(S^3), \tag{3.20}$$

where $f(t)$ satisfies the inequalities (3.4)-(3.6).

As shown in Section 3.1,

$$f'(t) \le M_\varepsilon(f) = \max\{1 - f^2 - \frac{m^{n-2}(1-m^2)}{f^{n-2}}, \varepsilon(m^2 - f^2)\}. \tag{3.21}$$

According Line (3.14) and (3.15), when $n = 3$, $\varepsilon = 0.2$, we have

$$\int_0^r f^2(t)dt \le \int_0^1 \frac{m^2 s^2 ds}{\sqrt{\max\{(1-s^2)[1-\frac{1-m^2}{m^2 s(1+s)}], 0.2(1-s^2)\}}}. \tag{3.22}$$

Since $\text{vol}(S^3) = 2\pi^2$, then to get inequality (3.20), we need to prove the following lemma.

Lemma 61. *When $0 \le m \le 1$, we have*

$$L(m) := \int_0^1 \frac{m^2 s^2 ds}{\sqrt{\max\{(1-s^2)[1-\frac{1-m^2}{m^2 s(1+s)}], 0.2(1-s^2)\}}} \le \frac{\pi}{4}. \tag{3.23}$$

Proof. First we assume

$$F(t) := \int_0^t \frac{s^2}{\sqrt{1-s^2}} ds = \frac{1}{2}\arcsin t - \frac{1}{2} t\sqrt{1-t^2}, \tag{3.24}$$

then $F(1) = \frac{\pi}{4}$. Therefore, if $m^2 \le \sqrt{0.2}$, then $L(m) \le \frac{\pi}{4}$. We can focus on $m^2 \ge \sqrt{0.2}$ for the rest of the proof.

Let $c \in (0,1)$, and we will pick c later. Denote

$$L_1(m) := \int_0^c \frac{m^2 s^2 ds}{[0.2(1-s^2)]^{1/2}},\ L_2(m) := \int_c^1 \frac{m^2 s^2 ds}{\sqrt{(1-s^2)[1-\frac{1-m^2}{m^2 s(1+s)}]}}. \tag{3.25}$$

Then $L_1(m) + L_2(m) \ge L(m)$. We will show that $L_1(m) + L_2(m) \le \frac{\pi}{4}$, after we carefully select c. We use different estimates, when m^2 lies in different intervals.

(1) $\varepsilon^{1/2} \le m^2 \le 0.5$

Let $c = 0.9$, then $L_1(m) = m^2\varepsilon^{-1/2}F(0.9) \le 0.5\varepsilon^{-1/2}F(0.9)$,

$$L_2(m) \le (F(1) - F(0.9)) \max_{\sqrt{0.2} \le m \le 0.5} \left\{ \frac{m^2}{\sqrt{1 - \frac{1-m^2}{m^2(0.9+0.9^2)}}} \right\}.$$

As $\frac{m^2}{\sqrt{1-\frac{1-m^2}{m^2(0.9+0.9^2)}}}$ is decreasing, when $0.2^{1/2} \le m^2 \le 0.5$, so plug $m^2 = 0.2^{1/2}$ into it, we have $\frac{m^2}{\sqrt{1-\frac{1-m^2}{m^2(0.9+0.9^2)}}} \le 0.85$.

Therefore, $L_1(m) + L_2(m) \le 0.5\varepsilon^{-1/2}F(0.9) + 0.85(F(1) - F(0.9)) \le 0.766 \le \frac{\pi}{4}$.

(2) $0.5 \le m^2 \le 0.6$

Let $c = 0.8$, $L_1(m) \le 0.6\varepsilon^{-1/2}g(0.8)$.

For the $L_2(m)$ part, we divide into two parts. We first integrate from 0.8 to 0.9, then integrate from 0.9 to 1.

Since $\frac{m^2}{\sqrt{1-\frac{1-m^2}{m^2(0.8+0.8^2)}}}$ is decreasing, when $m^2 \in [0.5, 0.6]$, then we have $\frac{m^2}{\sqrt{1-\frac{1-m^2}{m^2(0.8+0.8^2)}}} \le 1$. We also have $\frac{m^2}{\sqrt{1-\frac{1-m^2}{m^2(0.9+0.9^2)}}} \le 0.8$, when $m^2 \in [0.5, 0.6]$.

Hence, $L_2(m) \le F(0.9) - F(0.8) + 0.8(F(1) - F(0.9))$.

Then $L_1(m) + L_2(m) \le 0.778 \le F(1)$.

(3) $m^2 \ge 0.8$:

Let $c = c(m)$ satisfy $1 - \frac{1-m^2}{m^2(c+c^2)} = 0.2$, then c is well defined, since

$$c = \frac{-1 + \sqrt{5m^{-2} - 4}}{2} \le \frac{1}{4}.$$

When $t \in [c(m), 1]$, $1 - \frac{1-m^2}{m^2(c+c^2)} \ge 0.2$.

Assume $H(m) = \int_0^c \frac{m^2t^2}{[0.2(1-t^2)]^{1/2}}dt + \int_c^1 \frac{m^2t^2}{[(1-t^2)(1 - \frac{1-m^2}{m^2(t+t^2)})]^{1/2}}dt.$

Since the integrand is continuous at c, then when we compute $H'(m)$, we do not need to take the derivative of $c(m)$. Then we have

$$mH'(m) = \int_c^1 \frac{2m^2t^2}{\sqrt{(1-t^2)(1-\frac{1-m^2}{m^2(t+t^2)})}} - \frac{t(1+t)^{-1}}{(1-t^2)^{1/2}(1-\frac{1-m^2}{m^2(t+t^2)})^{3/2}}dt \tag{3.26}$$

$$+ \int_0^c \frac{2m^2t^2}{[0.2(1-t^2)]^{1/2}}dt \tag{3.27}$$

$$\geq 2m^2 \int_0^1 \frac{t^2dt}{\sqrt{1-t^2}} - \int_c^1 \frac{t(1+t)^{-1}}{(1-t^2)^{1/2}(1-\frac{1-m^2}{m^2(t+t^2)})^{3/2}}dt \tag{3.28}$$

$$\geq 0.4\pi - \int_0^1 \frac{tdt}{(1+t)(1-t^2)^{1/2}\max\{1-\frac{1-m^2}{m^2(t+t^2)}, 0.2\}^{3/2}} \tag{3.29}$$

$$\geq 0.4\pi - \int_0^1 \frac{tdt}{(1+t)(1-t^2)^{1/2}\max\{1-\frac{1}{4(t+t^2)}, 0.2\}^{3/2}} \tag{3.30}$$

$$\geq 1.256 - \sum_{i=1}^{20}[k(0.05i) - k(0.05(i-1))] \times w^{-3}(0.05(i-1)) \tag{3.31}$$

$$\geq 1.256 - 1.253, \tag{3.32}$$

in the second last expression, we assume $k(x) = \int_0^x \frac{tdt}{(1+t)(1-t^2)^{1/2}} = -1 + \frac{\sqrt{1-x^2}}{1+x} + \arcsin x$, $w(t) = \max\{1-\frac{1}{4(t+t^2)}, 0.2\}^{1/2}$. We use Mathematica to get the last inequality.

Therefore, $H(m)$ is increasing, $H(m) \leq H(1)$.

(4) $0.73 \leq m^2 \leq 0.8$:

Assume $l(t) = \max\{1 - \frac{0.27}{0.73(t+t^2)}, 0.2\}^{1/2}$, $c = \frac{-1+\sqrt{5m^{-2}-4}}{2} \geq 0.25$, and $w(t) = \max\{1 - \frac{1}{4(t+t^2)}, 0.2\}^{1/2}$.

Let us continue to estimate $mH'(m)$. According to line (3.26),

$$\begin{aligned} mH'(m) \geq & 2m^2 \int_0^1 \frac{t^2 dt}{\sqrt{1-t^2}w(t)} - \int_c^1 \frac{t(1+t)^{-1}}{(1-t^2)^{1/2}(1-\frac{1-m^2}{m^2(t+t^2)})^{3/2}} dt \\ \geq & 1.46 \sum_{i=2}^{20} [F(0.05i) - F(0.05(i-1))] \times [w(0.05i)]^{-1} \\ & - \sum_{i=26}^{100} [k(0.01i) - k(0.01(i-1))] \times [l(0.01(i-1))]^{-3} \\ \geq & 1.270 - 1.258. \end{aligned}$$

The last inequality is followed by Mathematica.

(5) $0.7 \leq m^2 \leq 0.73$:

We have $c(m) = \frac{-1+\sqrt{5m^{-2}-4}}{2} \geq 0.34$. Assume $p(t, m^2) = \max\{1 - \frac{1-m^2}{m^2(t+t^2)}, 0.2\}^{1/2}$, then $p(t, m^2)$ is decreasing with respect to t, and increasing with respect to m.

$$\begin{aligned} mH'(m) \geq & 2m^2 \int_0^1 \frac{t^2 dt}{\sqrt{1-t^2}p(t,m)} - \int_{0.34}^1 \frac{tdt}{(1+t)\sqrt{1-t^2}p^3(t,m)} \\ \geq & 1.4 \sum_{i=1}^{20} [F(0.05i) - F(0.05(i-1))] \times [p(0.05i, 0.73)]^{-1} \\ & - \sum_{i=35}^{100} [k(0.01i) - k(0.01(i-1))] \times [p(0.01(i-1), 0.7)]^{-3} \\ \geq & 1.291 - 1.212. \end{aligned}$$

The last inequality is followed by Mathematica.

(6) $0.68 \leq m^2 \leq 0.7$:

We have $c(m) = \frac{-1+\sqrt{5m^{-2}-4}}{2} \geq 0.38$.

$$
\begin{aligned}
mH'(m) \geq & 2m^2 \int_0^1 \frac{t^2 dt}{\sqrt{1-t^2}p(t,m)} - \int_{0.38}^1 \frac{tdt}{(1+t)\sqrt{1-t^2}p^3(t,m)} \\
\geq & 1.36 \sum_{i=1}^{100} [F(0.01i) - F(0.01(i-1))] \times [p(0.05i, 0.7)]^{-1} \\
& - \sum_{i=39}^{100} [k(0.01i) - k(0.01(i-1))] \times [p(0.01(i-1), 0.68)]^{-3} \\
\geq & 1.303 - 1.227.
\end{aligned}
$$

We use Mathematica to get this last inequality.

(7) $0.6 \leq m^2 \leq 0.68$:

Let $c = 0.5$. We have

$$
\int_0^{0.5} \frac{m^2 t^2 dt}{\sqrt{0.2(1-t^2)}} - \int_0^1 \frac{t^2 dt}{\sqrt{1-t^2}} \leq 0.68 \times 0.2^{-0.5} F(0.5) - F(1) \leq -0.71,
$$

and

$$
\begin{aligned}
\int_{0.5}^1 \frac{m^2 t^2 dt}{\sqrt{(1-t^2)(1-\frac{1-m^2}{m^2(t+t^2)})}} \leq & \sum_{i=51}^{100} 0.68[F(0.01i) - F(0.01(i-1))] \\
& \times p^{-1}(0.01(i-1), 0.6) \\
\leq & 0.681,
\end{aligned}
$$

where the last inequality is verified by Mathematica.

Hence, $L_1(m) + L_2(m) \leq F(1)$.

Then we finish the proof of this lemma. □

According to Lemma 61 and line 3.22, we have

$$\int_0^r f^2(t)dt \leq \frac{\pi}{4}. \tag{3.33}$$

Similarly, we have

$$\int_r^a f^2(t)dt \leq \frac{\pi}{4}. \tag{3.34}$$

Therefore,

$$\mathrm{vol}(M^3) = \int_0^a 4\pi f^2(t)dt \leq 2\pi^2 = \mathrm{vol}(S^3). \tag{3.35}$$

Finally, We conclude this appendix with the following theorem.

Theorem 62. *Let* (M^3, g) *be a three dimensional Riemannian manifold satisfying*

$$\mathrm{Ric}_g \geq 0.2g, \quad \text{and } R_g \geq n(n-1),$$

then $\mathrm{vol}(M^3) \leq \mathrm{vol}(S^3)$.

Chapter 4

Conclusion

We establish an lower bound for the ADM mass in asymptotically hyperbolic spaces.

Theorem. *Let (M, g, k) be a three dimensional, complete, simply connected asymptotically hyperbolic manifold a decay off rate $\tau > \frac{3}{2}$. Then*

$$E - P_i \geq \frac{1}{16\pi} \int_M \left[\frac{|\nabla^2 u + k|\nabla u||^2}{|\nabla u|} + 2(\mu|\nabla u| + \langle J, \nabla u \rangle) \right],$$

where E is the total energy, $P = (P_1, P_2, P_3)$ is the total momentum, and u satisfies the equation $\Delta u + \mathrm{tr}_g(k)|\nabla u| = 0$ with the asymptotic $u \to -t - x_i$ at infinity.

Note that (M, g, k) is simply connected in the theorem. To establish a positive mass theorem for manifolds with MOTs/MITs boundaries, we need to fill in the manifolds to reduce the inner boundaries. Although it is an active research topic to fill in a manifold with positive scalar curvature ([ST02, MMT19, MS15, Bar89, PCM18, Jau13]), there is little literature on the fill-ins in the spacetime setting.

Using the theorem above, we also prove two rigidity cases.

Theorem. *Let (M, g, k) be a three dimensional, complete, simply connected asymptotically hyperbolic manifold with a decay off rate $\tau > \frac{3}{2}$.*

1. *If $g = k$, and $E = |P|$, then (M, g) is isometric to hyperbolic space.*

2. *If $E = |P| = 0$, then (M, g) can be embedded in the Minkowski space.*

The general rigidity case $E = |P|$ remains open.

We also prove two theorems relate scalar curvature and volume. If the manifold is axisymmetric, we have

Theorem. *Let $n \geq 3$. If (M, g) is axisymmetric , i.e. $M = [0, a] \times_f S^{n-1}$, $g = dt^2 + f(t)^2 d\sigma^2$, where $d\sigma^2$ is the standard metric of S^{n-1}. There exists an $\varepsilon(n) < 1$, such that for any axisymmetric manifold (M, g) satisfies:*

$$\mathrm{Ric}(g) \geq \varepsilon(n) \cdot \mathrm{Ric}_0 \cdot g \text{ and } R(g) \geq R_0,$$

we have: $\mathrm{vol}(M) \leq \mathrm{vol}(S^n)$.

If we assume an upper bound on Ricci curvature, then we have a volume comparison theorem involving scalar curvature.

Theorem. *For any $C > 0$, there exists an $\varepsilon = \varepsilon(n, C) \in (0, 1)$, such that for any compact Riemannian manifold (M, g) satisfies*

1. $(1 - \varepsilon)(n - 1)g \leq \mathrm{Ric}_g \leq Cg$,

2. $R_g \geq n(n - 1)$,

then $\mathrm{vol}(M) \leq \mathrm{vol}(S^n)$.

We have proven that if the manifold has a given upper bound on Ricci curvature or the manifold is axis symmetric, then Bray's football theorem holds for dimensions larger than 3. These facts support us to believe that Bray's football theorem may be extended to all dimensions ($n \geq 3$). In addition to this problem, taking the diameter into consideration, we propose two questions.

Question 1. *Assume $\varepsilon \leq 1$, if (M, g) satisfies* $\mathrm{diam}(M) \leq \pi$, $\mathrm{Ric}_g \geq \varepsilon g$, *what is the sharp volume upper bound of M?*

According to Bishop theorem, M has a volume upper bound. However, the volume upper bound given by Bishop theorem may not optimal, except for $\varepsilon = \frac{1}{2}$. When $\varepsilon = \frac{1}{2}$, then M can be RP^n with $\mathrm{Ric}_g = \frac{1}{2}g$. If $\varepsilon \neq \frac{1}{2}$ and we assume the manifold has a upper bound on Ricci curvature, then according to [Wu95], the volume upper bound given by Bishop theorem is not optimal.

The second question is a volume comparison conjecture involving scalar curvature and diameter for positive Ricci curvature manifolds.

Question 2. *If* (M, g) *satisfies*

$$R_g \geq n(n-1),\ \mathrm{Ric}_g \geq 0,\ \mathrm{diam}(M) \leq \pi,$$

do we have: $\mathrm{vol}(M) \leq \mathrm{vol}(S^n)$*?*

We can not drop the positive Ricci condition in question 2, since a hyperbolic space products with S^1 may have a large volume.

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