Exam 1
Math 353
Summer Term I, 2021
Friday, May 28, 2021
Time Limit: 75 Minutes

Name: Solutions

This exam contains 7 pages (including this cover page) and 6 questions. The total number of points on this exam is 72. While this is a closed book, closed notes exam, you are allowed to use your one page review sheet, front and back, written on an 8.5 x 11 inch physical piece of paper in your own handwriting. While your review sheet must have been created with 20th century technology, you may take this exam on a tablet if you like, or on paper.

Show all of your work (on the backs of pages, if necessary) and circle your answers. If you do not completely solve a problem, explain what you do understand about it. If you would like to guess about a problem, clearly say so. This will maximize your partial credit. No collaboration on this exam is allowed. Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
Total:	72	

1. (12 points) Consider the differential equation where

$$y' + \frac{2}{t}\,y = \frac{6\cos(2t)}{t^2}.$$

(a) Sketch the slope field for this differential equation.

(b) Solve this differential equation for $t \geq \pi$ with initial condition $y(\pi) = 5$.

Multiply by t^2: $t^2y' + 2ty = 6\cos(2t)$

$$(t^2y)' = 6\cos(2t)$$

$$t^2y = 3\sin(2t) + C$$

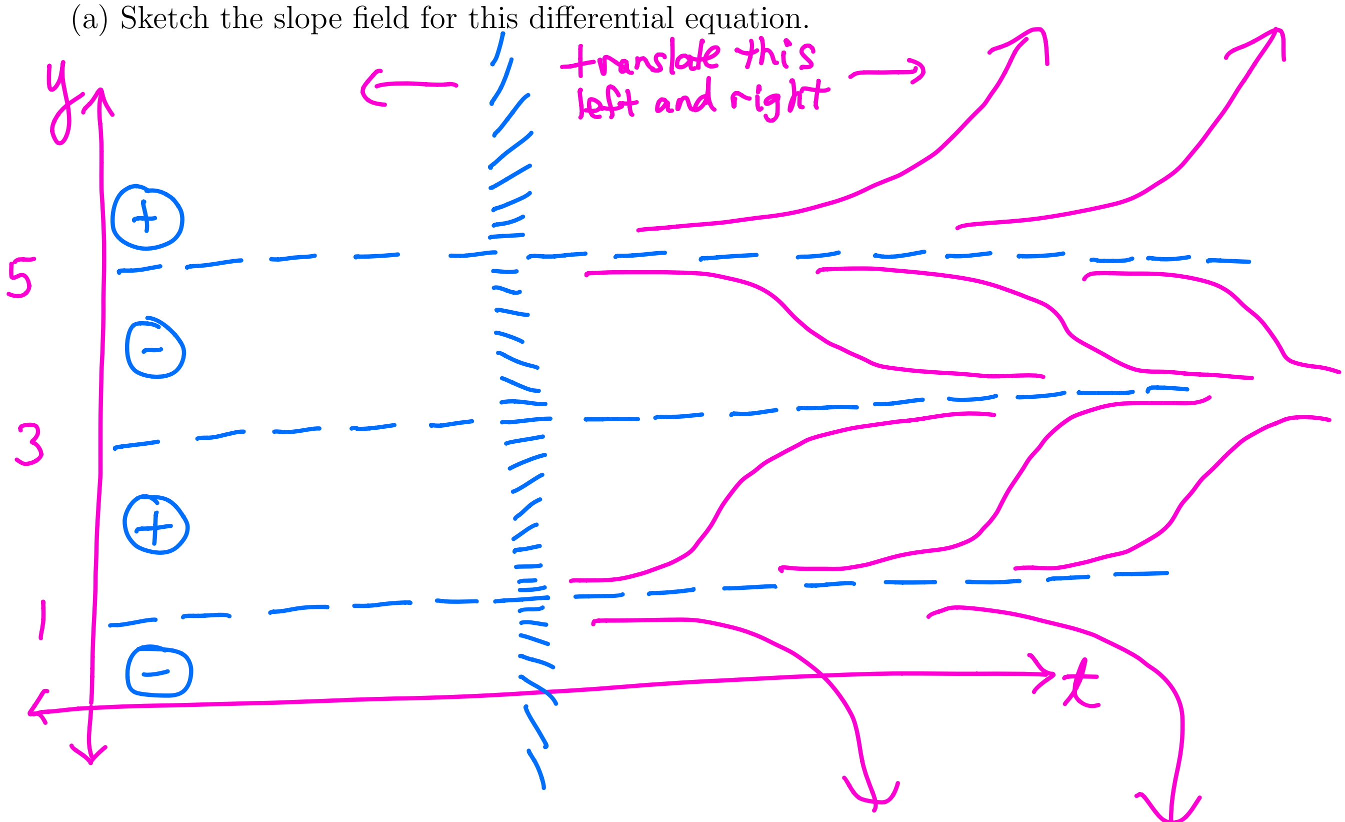
General Solution: $y = \dfrac{3\sin(2t) + C}{t^2}$

$y(\pi) = 5 \Longrightarrow$ $\boxed{y = \dfrac{3\sin(2t) + 5\pi^2}{t^2}}$

2. (12 points) Consider the differential equation where

$$\frac{dy}{dt} = (y-1)(3-y)(5-y).$$

(a) Sketch the slope field for this differential equation.



(b) If $y(0) = 4$, what is the behavior of $y(t)$ as t goes to infinity?

$$\lim_{t\to\infty} y(t) = 3$$

(c) If $y(0) = 2$, what is the behavior of $y(t)$ as t goes to infinity?

$$\lim_{t\to\infty} y(t) = 3$$

3. (12 points) A swinging door (that can open both to the inside and the outside) has both a spring and a dampener on it to control how well the door shuts. Define the angle of the swinging door at time t to be $\theta(t)$, where $\theta = 0$ corresponds to a shut door, $\theta > 0$ is opening to the outside, and $\theta < 0$ is opening to the inside.

$\theta'' + 2k\theta' + 4\theta = 0$

Suppose that the dynamics of the swinging door are controlled by

$r^2 + 2k \cdot r + 4 = 0$

$$\theta''(t) = -4\theta(t) - 2k\theta'(t).$$

$r = \frac{-2k \pm \sqrt{4k^2 - 16}}{2} = -k \pm \sqrt{k^2 - 4}$

In the above equation, the 4 represents the strength of the spring and the k represents the strength of the dampener (which resists motion in both directions).

$ar^2 + br + c = 0$

$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Find the general solution to the above differential equation when

(a) k = 3 $\quad r = -3 \pm \sqrt{5}$

$a = 1$
$b = 2k$
$c = 4$

door takes a longer time to close

$$\boxed{\theta(t) = c_1 e^{(-3+\sqrt{5})t} + c_2 e^{(-3-\sqrt{5})t}}$$

(b) k = 2 $\quad r = -2$

door closes fastest

$$\boxed{\theta(t) = e^{-2t}(c_1 + c_2 t)}$$

(c) k = 1 $\quad r = -1 \pm i\sqrt{3}$

door slams every time

$$\boxed{\theta(t) = e^{-t}\left(c_1 \cdot \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}t)\right)}$$

(d) Which of the above solutions represents an "optimally damped" swinging door, defined to be the case where the door may never hit the door jam at $\theta = 0$ (if we insert one) but otherwise has a general solution that closes as quickly as possible (meaning that for very large t, $\theta(t)$ is as small as possible)?

Since $2 > 3 - \sqrt{5}$, the solution in part (b) closes fastest without always slamming.

4. (12 points) (a) Find the general solution to the equation $y'' - 5y' + 6y = 0$.

$$y = c_1 e^{2t} + c_2 e^{3t}$$

$r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0$

$r = 2, 3$

(b) Using the method of undetermined coefficients, find the general solution to the differential equation $y'' - 5y' + 6y = 2e^t$.

Guess: $y = Ae^t$

$y' = Ae^t$

$y'' = Ae^t$

$Ae^t - 5Ae^t + 6Ae^t = 2e^t$

$A - 5A + 6A = 2$

$2A = 2$

$A = 1$

$$\boxed{y = c_1 e^{2t} + c_2 e^{3t} + e^t}$$

(c) Using variation of parameters, find the general solution to the differential equation $y'' - 5y' + 6y = 2e^t$.

$$y = -e^{2t} \cdot \int^t \frac{e^{3s} \cdot 2e^s}{e^{5s}} ds + e^{3t} \cdot \int^t \frac{e^{2s} \cdot 2e^s}{e^{5s}} ds$$

By Thm. 3.6.1

$y_1 = e^{2t}$ $\quad$ $y_2 = e^{3t}$

$y_1' = 2e^{2t}$ $\quad$ $y_2' = 3e^{3t}$

$W = e^{2t} \cdot 3e^{3t} - e^{3t} \cdot 2e^{2t}$

$W = e^{5t}$

$$y = -e^{2t} \int^t 2e^{-s} ds + e^{3t} \int^t 2e^{-2s} ds$$

$$= -e^{2t}(-2e^{-t} - c_1) + e^{3t}(-e^{-2t} + c_2)$$

$$= \boxed{e^t + c_1 e^{2t} + c_2 e^{3t}}$$

5. (12 points) Use the Laplace transform to solve $y'' - 2y' + 5y = 0$ where $y(0) = 3$ and $y'(0) = -3$.

$$0 = (s^2Y - 3s + 3) - 2(sY - 3) + 5Y$$

$$= (s^2 - 2s + 5)Y - 3s + 9$$

$$Y = \frac{3s - 9}{s^2 - 2s + 5} = \frac{3(s-1) - 3 \cdot 2}{(s-1)^2 + 2^2}$$

$$y(t) = 3e^t \cos(2t) - 3e^t \sin(2t)$$

6. (12 points) The goal of this problem is to find the general solution to the differential equation $ty'' - 2y' + 9t^5 y = 0$, where y is a function of t.

(a) Using the chain rule from single variable calculus, convert this to a differential equation where y is a function of x, where $x = t^3$.

$$x = t^3 \; ; \; t = x^{1/3}$$

$$\frac{dx}{dt} = 3t^2 = 3x^{2/3}$$

$$\frac{dy}{dt} = \frac{dx}{dt} \cdot \frac{dy}{dx} = 3x^{2/3} \frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dx}{dt} \cdot \frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= 3x^{2/3} \frac{d}{dx}\left(3x^{2/3} \frac{dy}{dx}\right) = 3x^{2/3}\left(3x^{2/3} \frac{d^2y}{dx^2} + 2x^{-1/3} \frac{dy}{dx}\right)$$

$$= 9x^{4/3} \cdot \frac{d^2y}{dx^2} + 6x^{1/3} \frac{dy}{dx}$$

Plug this into our diff. equation:

$$0 = x^{1/3}\left[9x^{4/3} \frac{d^2y}{dx^2} + \underbrace{6x^{1/3} \frac{dy}{dx}\right] - 2 \cdot 3x^{2/3} \frac{dy}{dx}}_{\text{cancel}} + 9x^{5/3} y$$

$$= 9x^{5/3}\left(\frac{d^2y}{dx^2} + y\right) \Rightarrow \boxed{\frac{d^2y}{dx^2} + y = 0} \quad (x \neq 0)$$

(b) Find the general solution to the differential equation for $y(x)$ you derived in part (a).

$$y = c_1 \cos(x) + c_2 \sin(x)$$

(c) What is the general solution to the original differential equation for $y(t)$?

$$y = c_1 \cos(t^3) + c_2 \sin(t^3)$$