Exam 2
Math 353
Summer Term I, 2021 **Name:** ______________________________
Friday, June 11, 2021
Time Limit: 75 Minutes

This exam contains 6 pages (including this cover page) and 5 questions. The total number of points on this exam is 60. While this is a closed book, closed notes exam, you are allowed to use your one page review sheet, front and back, written on an 8.5 x 11 inch physical piece of paper in your own handwriting. While your review sheet must have been created with 20th century technology, you may take this exam on a tablet if you like, or on paper.

Show all of your work (on the backs of pages, if necessary) and circle your answers. If you do not completely solve a problem, explain what you do understand about it. If you would like to guess about a problem, clearly say so. This will maximize your partial credit. No collaboration on this exam is allowed. Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
Total:	60	

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1. (12 points) Consider the differential equation

$$y''(t) + y(t) = \delta(t-\pi) - \delta(t-3\pi)$$

with initial conditions $y(0) = 0$ and $y'(0) = 0$.

(a) Compute the Laplace transform of both sides of the equation and solve for $Y(s)$.

$$s^2 Y(s) + Y(s) = e^{-\pi s} - e^{-3\pi s}$$

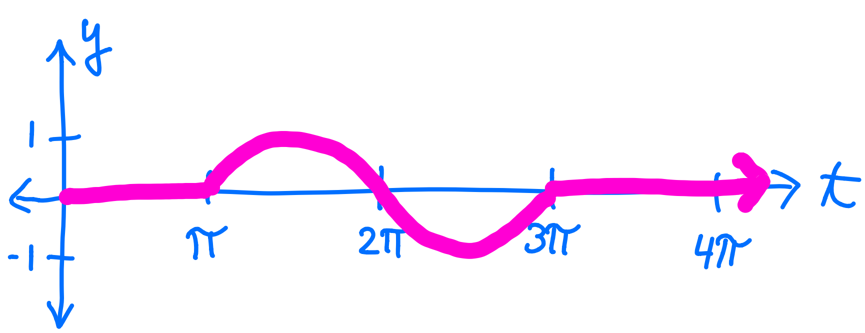
$$Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s^2+1}$$

(b) Compute $y(t)$ as the inverse Laplace transform of $Y(s)$.

$$y(t) = u_\pi(t)\sin(t-\pi) - u_{3\pi}(t)\sin(t-3\pi)$$

$$= \left(u_\pi(t) - u_{3\pi}(t)\right)\sin(t-\pi)$$

(c) Plot $y(t)$ for $0 \le t \le 4\pi$ and describe the behavior of $y(t)$ for large t.



$y(t) \equiv 0$ for $t \geq 3\pi$.

2. (12 points) Consider the harmonic function u defined in the unit disk $x^2 + y^2 \leq 1$ with boundary conditions $u = f(\theta)$ on the unit circle, where $f(\theta) = \sin(2\theta)$, and θ is the usual polar coordinate.

Recall that harmonic functions satisfy $u_{xx} + u_{yy} = 0$ in xy coordinates and $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ in polar coordinates.

(a) Compute $u(r, \theta)$ in the unit disk in polar coordinates.

$$u(r,\theta) = R(r)\Theta(\theta)$$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$$\underbrace{r^2\frac{R''}{R} + r\frac{R'}{R}}_{n^2} + \underbrace{\frac{\Theta''}{\Theta}}_{-n^2} = 0$$

$$\Theta'' + n^2\Theta = 0 \longrightarrow \Theta(\theta) = c_1\cos(n\theta) + c_2\sin(n\theta)$$

$$r^2R'' + rR' - n^2R = 0 \longrightarrow R(r) = \begin{cases} c_1 r^n + c_2 r^{-n}, & n \geq 1 \\ c_1 + c_2\ln(r), & n = 0 \end{cases}$$

by section 5.4 (Euler's equation)

Hence, we can match the given boundary conditions with

$$\boxed{u(r,\theta) = r^2\sin(2\theta)}$$

(b) Using the fact that $x = r\cos(\theta)$ and $y = r\sin(\theta)$, express u as a function of x and y.

$$u = r^2 \cdot 2\sin\theta\cos\theta$$

$$= 2(r\cos\theta)(r\sin\theta)$$

$$= \boxed{2xy}$$

(c) Verify that this function $u(x, y)$ is harmonic by computing $u_{xx} + u_{yy}$. What is the value of u when $x = 1/10$ and $y = 3/10$?

$$(2xy)_{xx} + (2xy)_{yy} = 0 + 0 = 0$$

$$u(x,y) = 2xy = 2\left(\frac{1}{10}\right)\left(\frac{3}{10}\right) = \frac{6}{100} = \boxed{\frac{3}{50}}$$

(or 0.06)

3. (12 points) Suppose a metal rod represented by the interval $0 \le x \le 1$ has an initial temperature of $u(x) = \sin\left(\frac{3\pi x}{2}\right)$ at $t = 0$. Suppose that $u(x,t)$ satisfies the heat equation

$L = 1$

$$u_t = u_{xx}$$

for $t \ge 0$, with boundary conditions $u(0,t) = 0$ (left end being kept at a temperature of zero) and $u_x(L,t) = 0$ (right end well insulated).

(a) Compute the temperature $u(x,t)$ of the metal rod for $t \ge 0$.

$u(x,t) = X(x)T(t) \rightarrow \frac{T'(t)}{T} = \frac{X''(x)}{X} = -\lambda$

$X''(x) + \lambda X = 0 \longrightarrow X(x) = \sin\left(\frac{3\pi}{2}x\right)$ satisfies this with $\lambda = \left(\frac{3\pi}{2}\right)^2$

$T'(t) = -\frac{9\pi^2}{4}T \rightarrow T(t) = c \cdot e^{-\frac{9\pi^2}{4}t} \longrightarrow$

$u(x,t) = e^{-\frac{9\pi^2}{4}t}\sin\left(\frac{3\pi}{2}x\right)$ solves $u_t = u_{xx}$ and satisfies the given initial and boundary conditions.

(b) What is the temperature of the metal rod at $x = 1$ when $t = 10$?

$u(1,10) = -e^{-\frac{45\pi^2}{2}}$

4. (12 points) Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x),$$

where

$$a_n = 2\int_0^1 x^3 \cos(n\pi x)\, dx. = \int_{-1}^{1} |x|^3 \cos(n\pi x)\, dx$$

Hence, $f(x) = |x|^3$ for $-1 \le x \le 1$ and has period 2.

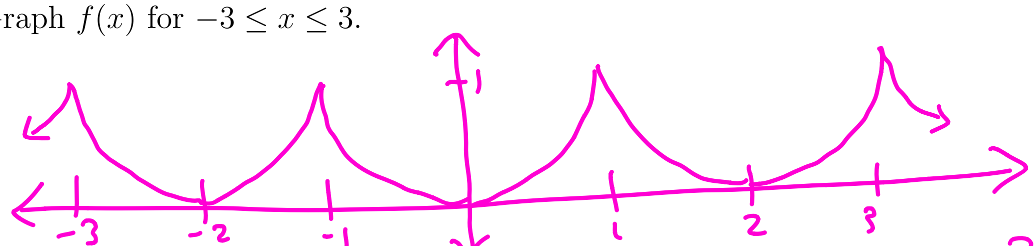
(a) Is $f(x)$ an even function, an odd function, or neither?

Even function, since it is the sum of cosines, which are even.

(b) What is the period of $f(x)$?

2, since that's the period of $\cos(\pi x)$, etc.

(c) Graph $f(x)$ for $-3 \le x \le 3$.



(d) What is $f(-5/2)$?

$$f(-5/2) = f\left(-\tfrac{5}{2}+2\right) = f\left(-\tfrac{1}{2}\right) = \left|-\tfrac{1}{2}\right|^3 = \tfrac{1}{8}.$$

5. (12 points) In this problem, you are NOT ALLOWED to use any sines or cosines. That would just make the problem harder anyway. In this problem, $-\infty < x < \infty$.

Consider the wave function $u(x,t)$ which is the sum of a left and right traveling wave:

$$u(x,t) = h(x+at) + k(x-at).$$

(a) Compute u_{tt} and u_{xx} in terms of h and k. Verify that $u(x,t)$ satisfies the wave equation $u_{tt} = a^2 u_{xx}$.

$u_t(x,t) = ah'(x+at) - ak'(x-at)$

$u_{tt}(x,t) = a^2h''(x+at) + a^2k''(x-at)$

$u_{tt} = a^2 u_{xx}$

$u_x(x,t) = h'(x+at) + k'(x-at)$

$u_{xx}(x,t) = h''(x+at) + k''(x-at)$

(b) Compute $u(x,0)$ and $u_t(x,0)$ in terms of the single variable functions h and k.

$u(x,0) = h(x) + k(x)$

$u_t(x,0) = a(h'(x) - k'(x))$

Plug in $t=0$.

(c) Suppose u has initial conditions

$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x)$$

for some smooth functions $f(x)$ and $g(x)$. Solve for the corresponding h and k which give these initial conditions. It is okay if your answer has a definite integral in it (which I suggest begins at zero, though it doesn't actually matter).

$$a(h(x) - k(x)) = \int_0^x a(h'(s) - k'(s))\,ds + C$$
$$= \int_0^x g(s)\,ds + C$$

$$h(x) - k(x) = \frac{1}{a}\int_0^x g(s)\,ds + \frac{C}{a}$$

$$h(x) + k(x) = f(x)$$

ADD to get $2h$, SUBTRACT to get $2k$.

$$h(x) = \frac{1}{2}f(x) + \frac{1}{2a}\int_0^x g(s)\,ds + \frac{C}{2a}$$
$$k(x) = \frac{1}{2}f(x) - \frac{1}{2a}\int_0^x g(s)\,ds - \frac{C}{2a}$$

(d) Using part (c), derive a formula for $u(x,t)$ in terms of f and g.

$$u(x,t) = \frac{1}{2}f(x+at) + \frac{1}{2a}\int_0^{x+at} g(s)\,ds + \frac{1}{2}f(x-at) - \frac{1}{2a}\int_0^{x-at} g(s)\,ds$$

$$u(x,t) = \frac{1}{2}\left[f(x+at) + f(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds$$