Exam 1
Math 353
Summer Term 1
May 27, 2022
Time Limit: 75 Minutes

Name: Key

This exam contains 6 pages (including this cover page) and 5 questions. The total number of points on this exam is 60. While this is a closed book, closed notes exam, you are allowed to use your one page review sheet, front and back, written on an 8.5 x 11 inch physical piece of paper in your own handwriting. While your review sheet must have been created with 20th century technology, you may take this exam on a tablet if you like, or on paper.

Show all of your work (on the backs of pages, if necessary) and circle your answers. If you do not completely solve a problem, explain what you do understand about it. If you would like to guess about a problem, clearly say so. This will maximize your partial credit. No collaboration on this exam is allowed. Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
Total:	60	

1. (12 points) (a) Find the general solution to the differential equation

$$(4t^2 + t^4)\frac{dy}{dt} + 2t^3 y = 4t^3$$

Standard form: $\frac{dy}{dt} + \frac{2t}{4+t^2} y = \frac{4t}{4+t^2}$

integrating factor: $e^{\int \frac{2t}{4+t^2}dt} = e^{\ln(4+t^2)} = 4+t^2$

$$(4+t^2)\frac{dy}{dt} + 2t \cdot y = 4t$$

$$\frac{d}{dt}\left[(4+t^2)y\right] = 4t$$

$$(4+t^2)y = 2t^2 + C$$

$$y = \frac{2t^2+C}{t^2+4}$$

(b) Find a solution where $y = 1$ when $t = 1$.

$$y = \frac{2t^2+3}{t^2+4}$$

2. (12 points) (a) Find an integrating factor (function of x) for the differential equation

$$(3x^3y + x^2y^2) + (x^4 + x^3y)\frac{dy}{dx} = 0$$

that makes the differential equation exact.

Divide thru by x:

$$\boxed{(3x^2y + xy^2) + (x^3 + x^2y)\frac{dy}{dx} = 0}$$

$$\downarrow \frac{\partial}{\partial y} \qquad \downarrow \frac{\partial}{\partial x}$$

$$3x^2 + 2xy = 3x^2 + 2xy \checkmark$$

(b) Find the general solution of the differential equation.

$$\frac{d}{dx}\left(x^3y + \frac{1}{2}x^2y^2\right) = 0$$

$$\boxed{x^3y + \frac{1}{2}x^2y^2 = C}$$

(c) Find the solution which has $y = 1$ when $x = 1$.

$$\boxed{x^3y + \frac{1}{2}x^2y^2 = \frac{3}{2}}$$

3. (12 points) (a) Find the general solution to the homogeneous differential equation

$$x^2y'' - 5xy' + 9y = 0 \quad , \; x > 0.$$

Let $y = x^r \rightarrow r(r-1) - 5r + 9 = 0$

$$r^2 - 6r + 9 = 0$$

$$(r-3)^2 = 0$$

$$r = 3, 3 \text{ (double root)}$$

$$\boxed{y = C_1 x^3 + C_2 x^3 \ln(x)}$$

(b) Find a particular solution to the differential equation

$$x^2y'' - 5xy' + 9y = x^5 \quad , \; x > 0.$$

Guess: $y = A \cdot x^5$

Then: $y' = 5A x^4$

$y'' = 20A x^3$

$\rightarrow 20A - 5 \cdot 5A + 9A = 1$

$4A = 1$

$A = 1/4$

$$\boxed{y = \frac{1}{4} x^5}$$

(c) Find the general solution to the differential equation

$$x^2y'' - 5xy' + 9y = x^5 \quad , \; x > 0.$$

$$\boxed{y = \frac{1}{4} x^5 + C_1 x^3 + C_2 x^3 \ln(x)}$$

4. (12 points) For this problem you may use the formulas in Table 6.2.1 that you copied into your review sheet, or work the problems directly. Compute the Laplace transform $F(s)$ of each of the following functions.

(a) $f(t) = e^{3t}$

$$F(s) = \frac{1}{s-3}$$

(Rule 2)

(b) $f(t) = t^2 e^{3t}$

$$F(s) = \frac{2}{(s-3)^3}$$

(c)

$$f(t) = \begin{cases} e^{3t} & \text{for } 0 \le t \le 4 \\ e^{4t} & \text{for } t > 4 \end{cases}$$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^4 e^{-st} e^{3t} + \int_4^\infty e^{-st} e^{4t}\,dt$$

$$= \int_0^4 e^{-(s-3)t}\,dt + \int_4^\infty e^{-(s-4)t}\,dt$$

$$= \frac{1}{s-3}\left(1 - e^{-4(s-3)}\right) + \frac{1}{s-4}\left(e^{-4(s-4)} - 0\right), \quad \text{for } s > 4$$

$$= \boxed{\frac{1}{s-3} + e^{16} \cdot e^{-4s} \frac{1}{s-4} - e^{12} e^{-4s} \cdot \frac{1}{s-3}}$$

5. (12 points) A swinging door (that can open both to the inside and the outside) has both a spring and a dampener on it to control how well the door shuts. Define the angle of the swinging door at time t to be $\theta(t)$, where $\theta = 0$ corresponds to a shut door, $\theta > 0$ is opening to the outside, and $\theta < 0$ is opening to the inside.

$\theta'' + 2k\theta' + 9\theta = 0$

Suppose that the dynamics of the swinging door are controlled by

$r^2 + 2kr + 9 = 0$

$$\theta''(t) = -9\theta(t) - 2k\theta'(t).$$

$\boxed{r = -k \pm \sqrt{k^2 - 9}}$

Typo! 9

In the above equation, the ~~4~~ 9 represents the strength of the spring and the k represents the strength of the dampener (which resists motion in both directions).

Find the general solution to the above differential equation when

(a) k = 3

$r = -3 \pm 0 = -3, -3$ (double root)

$\boxed{\text{Optimally damped}}$

$$\boxed{\theta(t) = c_1 e^{-3t} + \underbrace{c_2 t e^{-3t}}_{\text{larger term}}}$$

(b) k = 4

$r = -4 \pm \sqrt{7} = -4 + \sqrt{7}, -4 - \sqrt{7}$

$$\boxed{\theta(t) = c_1 e^{(-4-\sqrt{7})t} + \underbrace{c_2 e^{(-4+\sqrt{7})t}}_{\text{larger term}}}$$

(c) k = 5

$r = -5 \pm \sqrt{16} = -5 \pm 4 = -9, -1$

$$\boxed{\theta(t) = c_1 e^{-9t} + \underbrace{c_2 e^{-t}}_{\text{larger term}}}$$

(d) Of the three values of k listed above, which yields the most "optimally damped" swinging door, defined to be the case where the door may never hit the door jam at $\theta = 0$ (if we insert one) but otherwise has a general solution that closes as quickly as possible (meaning that for very large t, $\theta(t)$ is as small as possible)?

te^{-3t} decays faster than e^{-t} and $e^{(-4+\sqrt{7})t}$ since $3 > 1$ and $3 > 4 - \sqrt{7}$.

Also, $te^{-3t} > 0$, so the door never slams.

Hence solution (a) is the most optimally damped.