Final Exam
Math 353
Summer Term I, 2022
Thursday, June 23
Time Limit: 180 Minutes

Name: Solutions

This exam contains 10 pages (including this cover page) and 9 questions. The total number of points on this exam is 108. While this is a closed book, closed notes exam, you are allowed to use your one page review sheet, front and back, written on an 8.5 x 11 inch physical piece of paper in your own handwriting. While your review sheet must have been created with 20th century technology, you may take this exam on a tablet if you like, or on paper. You may also use your review sheets from the previous two exams.

Show all of your work (on the backs of pages, if necessary) and circle your answers. If you do not completely solve a problem, explain what you do understand about it. If you would like to guess about a problem, clearly say so. This will maximize your partial credit. No collaboration on this exam is allowed. Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
7	12	
8	12	
9	12	
Total:	**108**	

1. (12 points) Find the solutions to the following differential equations that are valid in a small neighborhood of the given initial conditions.

   (a) $y' = \frac{\sin(x)}{e^y}$ where $y = 1$ when $x = 0$.

$$e^y \, dy = \sin(x) \, dx$$

$$e^y = k - \cos(x) \longrightarrow k = e + 1$$

$$e^y = e + 1 - \cos(x)$$

$$\boxed{y = \ln(e + 1 - \cos(x))}$$

   (b) $\frac{dy}{dx} = y^{3/2}$ where $y(0) = 1$.

$$y^{-3/2} \, dy = dx$$

$$-2y^{-1/2} = x + C \longrightarrow C = -2$$

$$-2y^{-1/2} = x - 2$$

$$y^{1/2} = \frac{2}{2-x}$$

$$\boxed{y = \left(\frac{2}{2-x}\right)^2}$$

2. (12 points) Find the general solutions to the following differential equations.

(a) $ty' + 5y = t^2$

$$t^5 y' + 5t^4 y = t^6$$

$$\frac{d}{dt}(t^5 \cdot y) = t^6$$

$$t^5 \cdot y = \frac{1}{7}t^7 + C$$

$$\boxed{y = \frac{1}{7}t^2 + Ct^{-5}}$$

(b) $y'' - 4y' + 4y = 4e^{3t}$

$y = e^{rt} \rightarrow r^2 - 4r + 4 = 0$ for homogeneous problem.

$\rightarrow r = 2, 2$

$y_H = (C_1 + C_2 t)e^{2t}$

Guess $y_P = Ae^{3t} \rightarrow 9A - 4 \cdot 3A + 4A = 4$

$A = 4$

$y_P = 4e^{3t}$

$$\boxed{y = y_P + y_H = 4e^{3t} + (C_1 + C_2 t)e^{2t}}$$

3. (12 points) Consider the nonhomogeneous differential equation

$$y'' - 10y' + 25y = e^{5t}.$$

(a) What is the general solution to the corresponding homogeneous differential equation (right hand side equal to zero)?

$y = e^{rt} \rightarrow r^2 - 10r + 25 = 0 \rightarrow r = 5, 5$

$y_H = (c_1 + c_2 t)e^{5t}$

(b) Using the method of undetermined coefficients, find a particular solution to the original nonhomogeneous differential equation.

$2A = 1$

$A = 1/2$

Guess: $y_p = At^2 e^{5t}$

$y_p' = (5At^2 + 2At)e^{5t}$

$y_p'' = (25At^2 + 20At + 2A)e^{5t}$

$\boxed{y_p = \frac{1}{2}t^2 e^{5t}}$

(c) Find the solution to the original nonhomogeneous differential equation with $y(0) = 0$ and $y'(0) = 0$.

$y = \frac{1}{2}t^2 e^{5t} + (c_1 + c_2 t)e^{5t}$

$y' = (\frac{5}{2}t^2 + t)e^{5t} + (c_2 + 5c_1 + 5c_2 t)e^{5t}$

$0 = y(0) = c_1 \longrightarrow c_1 = 0 = c_2 \longrightarrow$

$0 = y'(0) = (c_2 + 5c_1)$

$\boxed{y = \frac{1}{2}t^2 e^{5t}}$

4. (12 points) Consider the differential equation

$$y''(t) + 6y'(t) + 5y(t) = 10 + \delta(t-2)$$

with initial conditions $y(0) = 0$ and $y'(0) = 0$.

(a) Compute the Laplace transform of both sides of the equation and solve for $Y(s)$.

$$(s^2+6s+5)Y(s) = \frac{10}{s} + e^{-2s}$$

$$Y(s) = \frac{10}{s(s+1)(s+5)} + \frac{e^{-2s}}{(s+1)(s+5)} = \ldots$$

use partial fractions

$$= \left(\frac{1/2}{s+5} - \frac{5/2}{s+1} + \frac{2}{s}\right) + \frac{1}{4}e^{-2s}\underbrace{\left(\frac{1}{s+1} - \frac{1}{s+5}\right)}_{H(s)}$$

$h(t) = e^{-t} - e^{-5t}$

(b) Compute $y(t)$ as the inverse Laplace transform of $Y(s)$.

$$y(t) = \frac{1}{2}e^{-5t} - \frac{5}{2}e^{-t} + 2 + \frac{1}{4}u_2(t)h(t-2)$$

$$= \frac{1}{2}e^{-5t} - \frac{5}{2}e^{-t} + 2 + \frac{1}{4}u_2(t)\left[e^{2-t} - e^{10-5t}\right]$$

(c) Describe the behavior of $y(t)$ for large t.

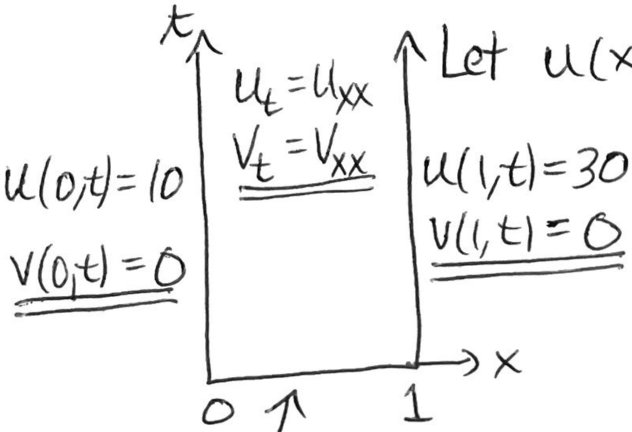
$\lim_{t\to\infty} y(t) = 2$ because all the other terms are decaying exponentials.

5. (12 points) Suppose a metal rod represented by the interval $0 \leq x \leq 1$ has an initial temperature of $u(x) = 10 + 20x + 5\sin(3\pi x)$ at $t = 0$. Suppose that $u(x,t)$ satisfies the heat equation

$$u_t = u_{xx}$$

for $t \geq 0$, with boundary conditions $u(0,t) = 10$ and $u(1,t) = 30$.

(a) Compute the temperature $u(x,t)$ of the metal rod for $t \geq 0$.



$u(x,0) = 10 + 20x + 5\sin(3\pi x)$
$v(x,0) = 5\sin(3\pi x)$

Let $u(x,t) = v(x,t) + (10 + 20x)$. Then

Sep. of Variables...

$v(x,t) = 5\sin(3\pi x) \cdot e^{-9\pi^2 t}$.

Hence,

$$u(x,t) = 10 + 20x + 5\sin(3\pi x)\, e^{-9\pi^2 t}$$

(b) What is the temperature of the metal rod at $x = 0.5$ when $t = 1$?

$$u(0.5, 1) = 10 + 10 + 5 \cdot (-1) \cdot e^{-9\pi^2}$$

$$= 20 - 5 \cdot e^{-9\pi^2}$$

6. (12 points) Let

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/2), \longrightarrow \text{odd, } L=2 \text{ period } = 4$$

where

$$a_n = 2\int_0^2 x^4 \sin(n\pi x/2)\,dx.$$

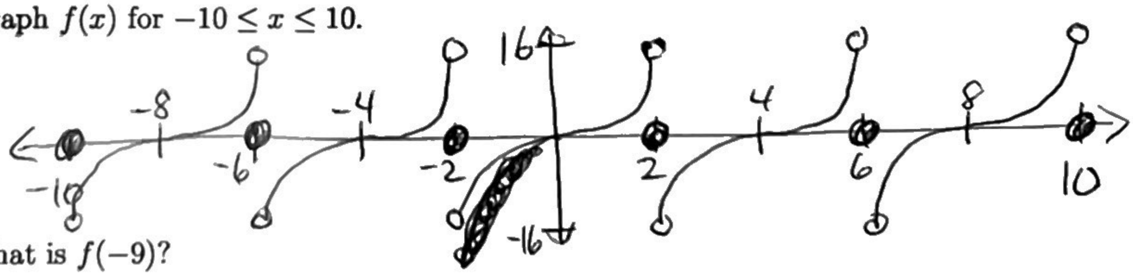
$$= \int_{-2}^{2} \left[\text{sign}(x) \cdot x^4\right] \sin\left(n\tfrac{\pi}{2}x\right) dx$$

$$= \begin{cases} x^4, & x \geq 0 \\ -x^4, & x \leq 0. \end{cases}$$

(a) Is $f(x)$ an even function, an odd function, or neither? What is the period of $f(x)$?

Odd, with period 4

(b) Graph $f(x)$ for $-10 \leq x \leq 10$.



(c) What is $f(-9)$?

$$f(-9) = f(-5) = f(-1) = -(-1)^4 = -1$$

(d) What is $f(6)$?

$f(6) = 0$ (Halfway between 16 and -16 as shown on graph)

7. (12 points) Solve the differential equation

$a_0 = 20$

$$4y'' + xy' - 2y = -3x$$

$a_1 = 3$

where y is expressed as a powers series in x around $x = 0$ in the case that $y(0) = 20$ and $y'(0) = 3$. Explicitly compute the first five terms of the power series.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$m = n-2$
$n = m+2$

$$xy' = \sum_{n=0}^{\infty} n a_n \cdot x^n$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n \cdot x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m$$

$$4y'' + xy' - 2y = \sum_{n=0}^{\infty} \left[4(n+2)(n+1)a_{n+2} + n a_n - 2a_n\right] x^n = -3x$$

Hence, $\boxed{4(n+2)(n+1)a_{n+2} + (n-2)a_n = \begin{cases} 0, & n \neq 1 \\ -3, & n = 1 \end{cases}}$

$$\boxed{a_{n+2} = \frac{2-n}{4(n+1)(n+2)} a_n \quad \text{for } n \neq 1}$$

$a_0 = 20$
$a_1 = 3$
$a_2 = \frac{2}{4 \cdot 1 \cdot 2} \cdot 20 = 5$
$a_3 = 0$
$a_4 = \frac{0}{4 \cdot 3 \cdot 4} \cdot 5 = 0$
$a_5 = 0$
$a_6 = 0$
$a_7 = 0$
$\vdots$

For $n = 1$,
$4 \cdot 3 \cdot 2 \cdot a_3 + (-1)a_1 = -3$
$a_1 = 3 \longrightarrow a_3 = 0.$

rest will be zero!

$\boxed{y = 20 + 3x + 5x^2}$ exactly!

8. (12 points) Suppose that a restaurant has an outdoor eating area (modeled as the xy plane) with a circular outdoor space heater of radius 1 (centered at the origin, modeled as the disk of radius 1). Suppose that the steady-state temperature $u(r, \theta)$ satisfies Laplace's equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,$$

goes to zero for large r, and that $u(1, \theta) = 20\cos(3\theta) - 30\sin(2\theta)$ on the boundary of the outdoor space heater.

(a) Solve for $u(r, \theta)$ for $r \geq 1$.

$$u(r,\theta) = 20r^{-3}\cos(3\theta) - 30r^{-2}\sin(2\theta)$$

goes to zero for large r, matches the boundary conditions, and is harmonic. From list:

1	$r^n\sin(n\theta)$	$r^n\cos(n\theta)$
$\ln r$	$r^{-n}\sin(n\theta)$	$r^{-n}\cos(n\theta)$

which comes from separation of variables.

(b) Suppose you are seated in the restaurant where $r = 5$ and $\theta = \pi/2$? What is the steady-state temperature at that location?

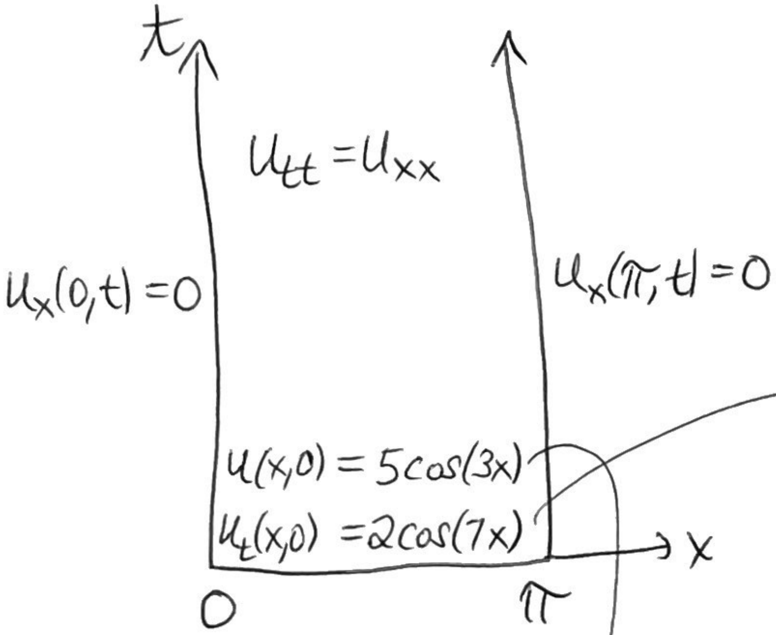
$$u(5, \tfrac{\pi}{2}) = 20 \cdot 5^{-3} \cdot 0 - 30 \cdot 5^{-2} \cdot 0 = \boxed{0}$$

9. (12 points) Suppose that $u(x,t)$ satisfies the wave equation

$$u_{tt} = u_{xx}$$

for $0 \le x \le \pi$ and $t \ge 0$, has Neumann boundary conditions $u_x(0,t) = 0$ and $u_x(\pi,t) = 0$, and initial conditions $u(x,0) = 5\cos(3x)$ and $u_t(x,0) = 2\cos(7x)$.

(a) Solve for $u(x,t)$.



sep. of variables

$$\boxed{u(x,t) = 5\cos(3x)\cos(3t) + \frac{2}{7}\cos(7x)\sin(7t)}$$

Note that

$$u_t(x,0) = 5\cos(3x)\cdot 0 + \frac{2}{7}\cdot\cos(7x)\cdot 7\cos(0)$$
$$= 2\cos(7x) \text{ as required.}$$

(b) What is $u(0,10)$?

$$u(0,10) = 5\cos(0)\cos(30) + \frac{2}{7}\cos(0)\sin(70)$$
$$= 5\cos(30) + \frac{2}{7}\sin(70).$$