Exam 1
Math 353
Summer Term 2
July 15, 2022
Time Limit: 75 Minutes

Name: Key

This exam contains 6 pages (including this cover page) and 5 questions. The total number of points on this exam is 60. While this is a closed book, closed notes exam, you are allowed to use your one page review sheet, front and back, written on an 8.5 x 11 inch physical piece of paper in your own handwriting.

Show all of your work (on the backs of pages, if necessary) and circle your answers. If you do not completely solve a problem, explain what you do understand about it. If you would like to guess about a problem, clearly say so. This will maximize your partial credit. No collaboration on this exam is allowed. Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
Total:	60	

1. (12 points) (a) Find the general solution for $t > 0$ to the differential equation

$$\frac{dy}{dt} + \frac{3}{t}y = 6t^2$$

$$t^3\frac{dy}{dt} + 3t^2 y = 6t^5$$

$$\frac{d}{dt}(t^3 y) = 6t^5$$

$$t^3 y = t^6 + C$$

$$\boxed{y = t^3 + \frac{C}{t^3}}$$

(b) Find the solution to the differential equation

$$\frac{dy}{dx} = \frac{3x^2 + 8x^3}{2y + 15y^4}$$

that has $y = 1$ when $x = 1$, for x values in some interval around $x = 1$.

$$(2y + 15y^4)dy = (3x^2 + 8x^3)dx$$

$$y^2 + 3y^5 = x^3 + 2x^4 + C$$

$$1 + 3 = 1 + 2 + C \longrightarrow C = 1$$

$$\boxed{y^2 + 3y^5 = x^3 + 2x^4 + 1}$$

2. (12 points) Suppose that the population density $y(t)$ of a certain type of bird in the United States is described by the differential equation

$$y'(t) = \frac{(y-50)(100-y)y}{5000}.$$

(a) Suppose $y(0) = 75$. Without explicitly solving the above differential equation, what will $y(t)$ converge to for very large t?

$y'(t) > 0$ until $y = 100$, so $\boxed{100}$.

(b) Suppose $y(0) = 25$. Without explicitly solving the above differential equation, what will $y(t)$ converge to for very large t?

$y'(t) < 0$ until $y = 0$, so $\boxed{0}$.

Now suppose that the population density $z(t)$ of certain type of squirrel in the United States is described by the differential equation

$$z'(t) = \frac{(100-z)z}{100}.$$

(c) Suppose $z(0) = 25$. Without explicitly solving the above differential equation, what will $z(t)$ converge to for very large t?

$z'(t) > 0$ until $z = 100$, so $\boxed{100}$.

(d) Which animal, the bird or the squirrel, has the greater need for protection from over hunting, or are they equally vulnerable? Points will only be award if the reason behind your answer is clearly explained.

The bird has the greater need for protection from over hunting since $y < 50$ leads to extinction for the bird, whereas the squirrel does not go extinct as long as $z > 0$.

3. (12 points) (a) Find the general solution to the homogeneous differential equation

$$x^2y'' - 6xy' + 10y = 0 \quad , \; x > 0.$$

Euler equation, so guess $y = x^r$. $\rightarrow$ $r(r-1) - 6r + 10 = 0$

$r^2 - 7r + 10 = 0$

$(r-2)(r-5) = 0$

$r = 2, 5$

$$\boxed{y = C_1 x^2 + C_2 x^5}$$

(b) Find a particular solution to the differential equation

$$x^2y'' - 6xy' + 10y = 12x^3 \quad , \; x > 0.$$

Guess: $y = Ax^3$

$y' = 3Ax^2$

$y'' = 6Ax$

$\rightarrow 6Ax^3 - 18Ax^3 + 10Ax^3 = 12x^3$

$-2A = 12$

$A = -6$

$$\boxed{y = -6x^3}$$

(b) Find a particular solution to the differential equation

(Guess: $y = Ax^5$ does not work)

$$x^2y'' - 6xy' + 10y = 12x^5 \quad , \; x > 0.$$

• $x^5\ln(x)$ terms cancel

Guess: $y = Ax^5\ln(x)$

$y' = A(5x^4\ln(x) + x^4)$

$y'' = A(20x^3\ln(x) + 5x^3 + 4x^3)$

$\rightarrow A[x^2(9x^3) - 6x(x^4)] = 12x^5$

$3A = 12$

$A = 4$

$$\boxed{y = 4x^5\ln(x)}$$

4. (12 points) For this problem you may use the formulas in Table 6.2.1 that you copied into your review sheet, or work the problems directly. Compute the Laplace transform $F(s)$ of each of the following functions.

(a) $f(t) = \sin 3t$

$$\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$$

(b) $f(t) = t^2 + t$

$$\mathcal{L}\{t^2+t\} = \mathcal{L}(t^2) + \mathcal{L}(t) = \frac{2}{s^3} + \frac{1}{s^2}$$

(c)

$$f(t) = \begin{cases} 0 & \text{for } 0 \le t \le 10 \\ e^{4(t-10)} & \text{for } t > 10 \end{cases}$$

$f(t) = u_{10}(t)\, g(t-10)$, where

$g(t) = e^{4t}$

$\downarrow$

$G(s) = \dfrac{1}{s-4}$

Rule $\downarrow$ 13

$$F(s) = e^{-10s} \cdot G(s)$$

$$\boxed{F(s) = \frac{e^{-10s}}{s-4}}$$

5. (12 points) A swinging door (that can open both to the inside and the outside) has both a spring and a dampener on it to control how the door shuts. Define the angle (in degrees) of the swinging door at time t (in minutes) to be $\theta(t)$, where $\theta = 0$ corresponds to a shut door, $\theta > 0$ is opening to the outside, and $\theta < 0$ is opening to the inside.

Suppose that the dynamics of the swinging door are controlled by

$$\theta''(t) = -6\theta(t) - 5\theta'(t).$$

In the above equation, the 6 represents the strength of the spring and the 5 represents the strength of the dampener (which resists motion in both directions).

(a) Find the general solution to the above differential equation.

$$\theta'' + 5\theta' + 6\theta = 0$$
$$r^2 + 5r + 6 = 0$$
$$(r+2)(r+3) = 0$$
$$r = -2, -3 \rightarrow \boxed{\theta(t) = c_1 e^{-2t} + c_2 e^{-3t}}$$

(b) Find the solution to the above differential equation if $\theta(0) = 50$ and $\theta'(0) = 0$. This corresponds to a person walking through a door and then releasing the door with no initial angular velocity.

$$\theta(0) = c_1 + c_2 = 50$$
$$\theta'(0) = -2c_1 - 3c_2 = 0$$
$$\rightarrow c_1 = 150, \quad c_2 = -100$$

$$\theta(t) = 150e^{-2t} - 100e^{-3t}$$

(c) Suppose $\theta(0) = 50$ but that you can choose $\theta'(0)$ to be whatever you want by pushing on the door sufficiently hard. What should you choose $\theta'(0)$ to be to get the door to be as closed as possible for large t?

Want to "zero out" the e^{-2t} term which goes to zero slower than the e^{-3t} term.

$$y = Ae^{-3t} \longrightarrow y(0) = A \longrightarrow A = 50$$
$$y' = -3Ae^{-3t} \longrightarrow y'(0) = -3A$$

$$\boxed{y'(0) = -150}$$