Exam 2
Math 353
Summer Term 2
Friday, July 29, 2022
Time Limit: 75 Minutes

Name: Key

This exam contains 6 pages (including this cover page) and 5 questions. The total number of points on this exam is 60. While this is a closed book, closed notes exam, you are allowed to use your one page review sheet for this exam (as well as the one from the first exam) written on an 8.5 x 11 inch physical piece of paper (front and back) in your own handwriting.

Show all of your work (on the backs of pages, if necessary) and circle your answers. If you do not completely solve a problem, explain what you do understand about it. If you would like to guess about a problem, clearly say so. This will maximize your partial credit. No collaboration on this exam is allowed. Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
Total:	60	

1. (12 points) Consider the differential equation

$$y''(t) + y(t) = t + \delta(t - 2\pi)$$

with initial conditions $y(0) = 0$ and $y'(0) = 0$.

(a) Compute the Laplace transform of both sides of the equation and solve for $Y(s)$.

$$(s^2+1)Y(s) = s^2 Y(s) + Y(s) = \frac{1}{s^2} + e^{-2\pi s}$$

$$\boxed{Y(s) = \frac{1}{s^2(s^2+1)} + \frac{e^{-2\pi s}}{s^2+1}}$$

$$= \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1}$$

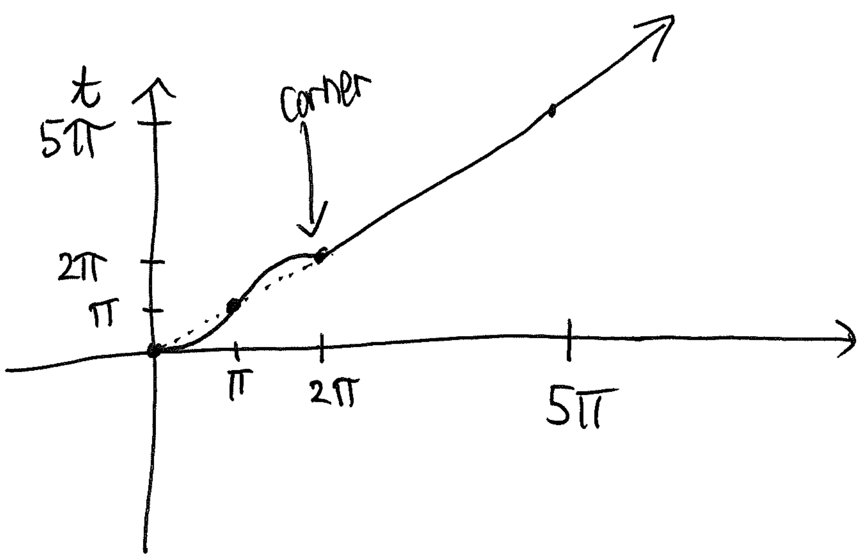
(b) Compute $y(t)$ as the inverse Laplace transform of $Y(s)$.

$$y(t) = t - \sin(t) + u_{2\pi}(t)\sin(t-2\pi)$$

$$= t - \sin(t) + u_{2\pi}(t)\sin(t)$$

$$= t + \sin(t)(u_{2\pi}(t) - 1) = \begin{cases} t - \sin t, & 0 < t < 2\pi \\ t, & t \geq 2\pi. \end{cases}$$

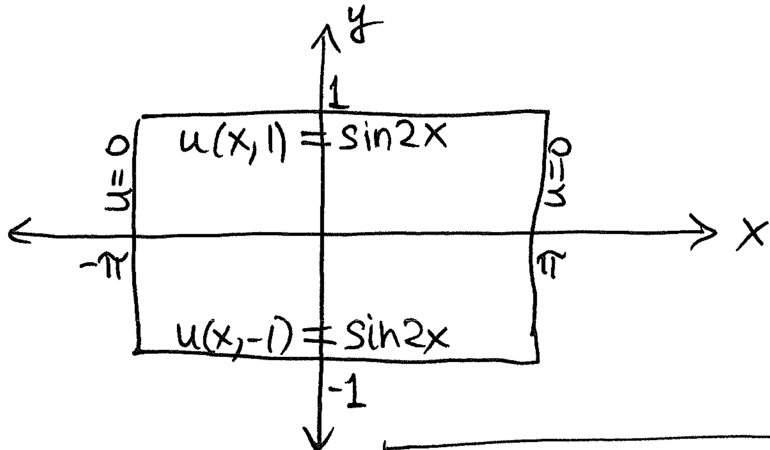
(c) Plot $y(t)$ for $0 \leq t \leq 5\pi$. What is y(100)?



$y(100) = 100.$

2. (12 points) Solve for the harmonic function $u(x, y)$ defined in the region $-\pi \le x \le \pi$ and $-1 \le y \le 1$ which has boundary conditions

$$u(-\pi, y) = 0 = u(\pi, y), \text{ for } -1 \le y \le 1 \text{ and}$$
$$u(x, -1) = \sin(2x) = u(x, 1), \text{ for } -\pi \le x \le \pi.$$



Separation of variables: $u(x,y) = X(x)Y(y)$

$$0 = u_{xx} + u_{yy}$$

$$0 = X''(x)Y(y) + X(x)Y''(y)$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda \Rightarrow \begin{array}{l} ① \; X''(x) + \lambda X = 0, \; X(-\pi) = 0 = X(\pi) \\ ② \; Y''(y) - \lambda Y = 0 \end{array}$$

so that

$X(x) = \sin(2x)$ is a solution with $\lambda = 4 \Rightarrow$

$Y(y) = c_1 \cosh(2y) + c_2 \sinh(2y)$.

$\left(\begin{array}{l} c_1 = \frac{1}{\cosh 2} \\ c_2 = 0 \end{array} \right)$

Matching up boundary conditions, we see that

$$u(x,y) = X(x)Y(y) = \boxed{\frac{\sin(2x)\cosh(2y)}{\cosh(2)}}.$$

solves the problem.

3. (12 points) Suppose a metal rod represented by the interval $0 \le x \le 1$ has an initial temperature of $u(x) = 20 + 10\cos(3\pi x)$ at $t = 0$. Suppose that $u(x,t)$ satisfies the heat equation

$$9u_t = u_{xx}$$

for $t \ge 0$, with boundary conditions $u_x(0,t) = 0$ (left end well insulated) and $u_x(1,t) = 0$ (right end well insulated).

(a) Compute the temperature $u(x,t)$ of the metal rod for $t \ge 0$.

$$u(x,t) = X(x)T(t) \longrightarrow 9X \cdot T'(t) = X''(x) \cdot T \rightarrow \frac{9T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

① $X''(x) + \lambda X = 0$, $X'(0) = 0 = X'(1) \longrightarrow X(x) = \cos(n\pi x)$, $\lambda = n^2\pi^2$

② $9T'(t) + \lambda T = 0$, $T(0) = 1 \longrightarrow T(t) = e^{-\frac{n^2\pi^2}{9}t}$ so that

$$u_n(x,t) = \cos(n\pi x)\, e^{-\frac{n^2\pi^2}{9}t}$$

Note that $u_0(x,t) = 1$ and $u_3(x,t) = \cos(3\pi x)e^{-\pi^2 t}$

so that we can match the initial conditions with

$$\boxed{u(x,t) = 20 + 10\cos(3\pi x)e^{-\pi^2 t}}$$

(b) What is the temperature of the metal rod at $x = 0$ when $t = 1$?

$$u(0,1) = 20 + 10e^{-\pi^2}$$

4. (12 points) Let

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x), \longrightarrow \text{odd}, L=1, \text{ period}=2$$

where

$$a_n = 20 \int_0^1 x^4 \sin(n\pi x)\, dx.$$

$$= \frac{1}{L}\int_{-L}^{L} 10 \underbrace{\text{sgn}(x) \cdot x^4}_{} \cdot \overbrace{\sin(n\pi x)}^{\text{odd}}\, dx$$

$$\boxed{f(x) = 10 \begin{cases} -x^4, & x \le 0 \\ x^4, & x \ge 0 \end{cases} \text{ is } \underline{\text{odd}}}$$

$$= \frac{1}{L}\int_{-L}^{L} f(x) \sin(n\pi x)\, dx$$

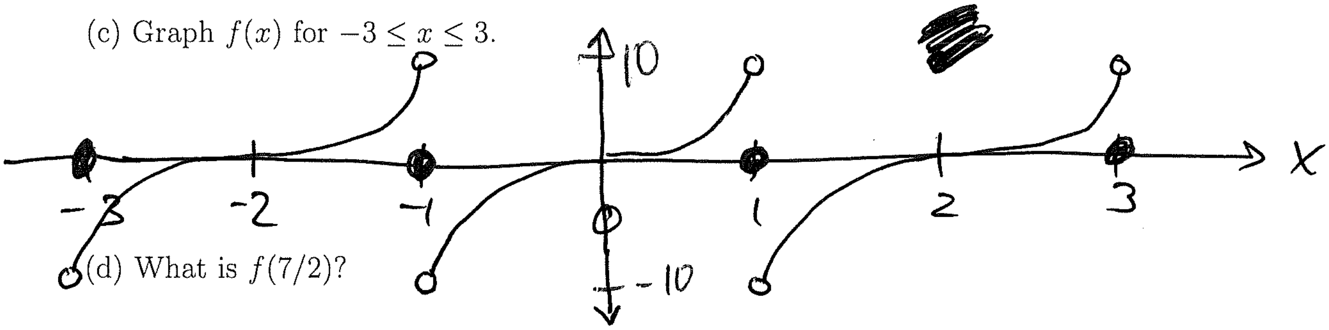
(a) Is $f(x)$ an even function, an odd function, or neither?

Odd

(b) What is the period of $f(x)$?

2

(c) Graph $f(x)$ for $-3 \le x \le 3$.



(d) What is $f(7/2)$?

$$f\left(\tfrac{7}{2}\right) = f\left(\tfrac{7}{2} - 4\right) = f\left(-\tfrac{1}{2}\right) = -\left(\tfrac{1}{2}\right)^4 \cdot 10 = -\frac{10}{16}$$

$$= -5/8$$

5. (12 points) Suppose a cube-shaped speaker box encloses the three dimensional region $0 \le x, y, z \le \pi$ and that $u(x, y, z, t)$, representing the vibrations of air molecules inside the speaker box, satisfies the wave equation

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}$$ wave equation

with boundary conditions

$u_x = 0$ on the two sides of the box where x equals a constant,

$u_y = 0$ on the two sides of the box where y equals a constant, and

$u_z = 0$ on the two sides of the box where z equals a constant.

If $u_t(x, y, z, 0) = 0$ and

$$u(x, y, z, 0) = \cancel{\cos(3x)\cos(4x)\cos(12x),}$$ typo in problem

$$= \cos(3x)\cos(4y)\cos(12z)$$

find $u(x, y, z, t)$ for all $t \ge 0$.

$$u(x,y,z,t) = X(x)Y(y)Z(z)T(t) \rightarrow \frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)}$$

from wave equation divided by $XYZT$.

$$\frac{X''(x)}{X(x)} = -3^2 \rightarrow X(x) = \cos(3x)$$

$$\frac{Y''(y)}{Y(y)} = -4^2 \rightarrow Y(y) = \cos(4y)$$

$$\frac{Z''(z)}{Z(z)} = -12^2 \rightarrow Z(z) = \cos(12z)$$

add

$$\frac{T''(t)}{T(t)} = -3^2 - 4^2 - 12^2 = -169 = -13^2 \rightarrow T(t) = c_1\cos(13t) + c_2\sin(13t)$$

Then $T'(t) = 0 \rightarrow c_2 = 0 \rightarrow$

$T(t) = 1 \rightarrow c_1 = 0$

$$\boxed{u(x,y,z,t) = \cos 3x \cdot \cos 4y \cdot \cos 5z \cdot \cos 13t}$$

Bonus ~~6~~. (12 points) Suppose a cube-shaped speaker box encloses the three dimensional region $0 \le x, y, z \le \pi$ and that $u(x, y, z, t)$, representing the vibrations of air molecules inside the speaker box, satisfies the wave equation

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}$$

with boundary conditions

$u_x = 0$ on the two sides of the box where x equals a constant,

$u_y = 0$ on the two sides of the box where y equals a constant, and

$u_z = 0$ on the two sides of the box where z equals a constant.

If $u_t(x, y, z, 0) = 0$ and

$$u(x, y, z, 0) = \cos(3x)\cos(4x)\cos(12x),$$

find $u(x, y, z, t)$ for all $t \ge 0$.

$$\cos(7x) = \cos(4x + 3x) = \cos 4x \cos 3x - \sin 4x \sin 3x$$
$$\cos(x) = \cos(4x - 3x) = \cos 4x \cos 3x + \sin 4x \sin 3x$$
$$\tfrac{1}{2}(\cos(7x) + \cos(x)) = \cos(4x)\cos(3x)$$

Similarly,

$$\cos(3x)\cos(4x)\cos(12x) = \tfrac{1}{2}(\cos 7x + \cos x)\cos 12x$$
$$= \tfrac{1}{2}\left[\tfrac{1}{2}(\cos 19x + \cos 5x) + \tfrac{1}{2}(\cos 13x + \cos 11x)\right]$$
$$= \tfrac{1}{4}\left[\cos 5x + \cos 11x + \cos 13x + \cos 19x\right]$$

Hence, we have the solution

$$u(x,y,z,t) = \tfrac{1}{4}\left[\cos 5x \cos 5t + \cos 11x \cos 11t + \cos 13x \cos 13t + \cos 19x \cos 19t\right]$$