

Exam 2
Math 353
Summer Term 2
Friday, July 29, 2022
Time Limit: 75 Minutes

Name: _____

This exam contains 6 pages (including this cover page) and 5 questions. The total number of points on this exam is 60. While this is a closed book, closed notes exam, you are allowed to use your one page review sheet for this exam (as well as the one from the first exam) written on an 8.5 x 11 inch physical piece of paper (front and back) in your own handwriting.

Show all of your work (on the backs of pages, if necessary) and circle your answers. If you do not completely solve a problem, explain what you do understand about it. If you would like to guess about a problem, clearly say so. This will maximize your partial credit. No collaboration on this exam is allowed. Good luck!

Grade Table (for teacher use only)

| Question | Points | Score |
|----------|--------|-------|
| 1 | 12 | |
| 2 | 12 | |
| 3 | 12 | |
| 4 | 12 | |
| 5 | 12 | |
| Total: | 60 | |

1. (12 points) Consider the differential equation

$$y''(t) + y(t) = t + \delta(t - 2\pi)$$

with initial conditions $y(0) = 0$ and $y'(0) = 0$.

- (a) Compute the Laplace transform of both sides of the equation and solve for $Y(s)$.

- (b) Compute $y(t)$ as the inverse Laplace transform of $Y(s)$.

- (c) Plot $y(t)$ for $0 \leq t \leq 5\pi$. What is $y(100)$?

2. (12 points) Solve for the harmonic function $u(x, y)$ defined in the region $-\pi \leq x \leq \pi$ and $-1 \leq y \leq 1$ which has boundary conditions

$$u(-\pi, y) = 0 = u(\pi, y), \text{ for } -1 \leq y \leq 1 \text{ and}$$

$$u(x, -1) = \sin(2x) = u(x, 1), \text{ for } -\pi \leq x \leq \pi.$$

3. (12 points) Suppose a metal rod represented by the interval $0 \leq x \leq 1$ has an initial temperature of $u(x) = 20 + 10 \cos(3\pi x)$ at $t = 0$. Suppose that $u(x, t)$ satisfies the heat equation

$$9u_t = u_{xx}$$

for $t \geq 0$, with boundary conditions $u_x(0, t) = 0$ (left end well insulated) and $u_x(1, t) = 0$ (right end well insulated).

- (a) Compute the temperature $u(x, t)$ of the metal rod for $t \geq 0$.

- (b) What is the temperature of the metal rod at $x = 0$ when $t = 1$?

4. (12 points) Let

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x),$$

where

$$a_n = 20 \int_0^1 x^4 \sin(n\pi x) dx.$$

(a) Is $f(x)$ an even function, an odd function, or neither?

(b) What is the period of $f(x)$?

(c) Graph $f(x)$ for $-3 \leq x \leq 3$.

(d) What is $f(7/2)$?

5. (12 points) Suppose a cube-shaped speaker box encloses the three dimensional region $0 \leq x, y, z \leq \pi$ and that $u(x, y, z, t)$, representing the vibrations of air molecules inside the speaker box, satisfies the wave equation

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}$$

with boundary conditions

$u_x = 0$ on the two sides of the box where x equals a constant,

$u_y = 0$ on the two sides of the box where y equals a constant, and

$u_z = 0$ on the two sides of the box where z equals a constant.

If $u_t(x, y, z, 0) = 0$ and

$$u(x, y, z, 0) = \cos(3x) \cos(4y) \cos(12z),$$

find $u(x, y, z, t)$ for all $t \geq 0$.