Exam 1
Math 353
Summer Term I, 2014
Friday, May 30, 2014
Time Limit: 75 Minutes

Name: Solutions

This exam contains 6 pages (including this cover page) and 6 questions. The total number of points on this exam is 72.

Your are allowed to use a calculator on this exam, though it is not really necessary. While this is a closed book exam, you are allowed to use your one page review sheet, front and back, written in your own handwriting.

Show all of your work (on the backs of pages, if necessary) and circle your answers. If you do not completely solve a problem, explain what you do understand about it. If you would like to guess about a problem, clearly say so. This will maximize your partial credit. No collaboration on this exam is allowed. Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
Total:	72	

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1. (12 points) Find a second order differential equation, with specified initial conditions, for which $y(t) = t^2 + t + 1$ is the **unique solution.** There are many correct answers.

$$\begin{cases} y''(t) = 2 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$

(since $y'(t) = 2t + 1$)

2. (12 points) Consider the initial value problem

$$y' = y^2, \qquad y(6) = 1/10.$$

(a) Does a unique solution to this initial value problem, in a neighborhood of the initial conditions, exist? Justify your answer by using the appropriate theorem from class.

$y' = f(t,y) = y^2 \leftarrow$ both are continuous $\Rightarrow$ local existence and uniqueness $\Rightarrow$ Yes

$\frac{\partial f}{\partial y} = 2y \leftarrow$

(b) Find the solution $y(t)$ to this initial value problem.

$$\frac{dy}{dt} = y^2 \rightarrow \frac{dy}{y^2} = dt \rightarrow \int \frac{dy}{y^2} = \int dt + c$$

$$\rightarrow -\frac{1}{y} = t + c \rightarrow -10 = 6 + c \rightarrow c = -16 \rightarrow$$

$$-\frac{1}{y} = t - 16 \rightarrow y = \frac{1}{16-t}$$

(c) What is the interval of existence for the solution your found? That is, what is the maximal interval on which your solution solves the given initial value problem?

$$-\infty < t < 16$$

3. (12 points) Find the general solutions to the following differential equations.

(a) $ty' - 2y = 8t^6$ $\longrightarrow$ $y' - \frac{2}{t}y = 8t^5$

Integrating factor: $e^{\int -2/t \, dt} = e^{-2\ln t} = t^{-2} \longrightarrow$

$$t^{-2}y' - 2t^{-3}y = 8t^3$$

$$\frac{d}{dt}\left(t^{-2}y\right) = 8t^3$$

$$t^{-2}y = 2t^4 + c$$

$$\boxed{y = 2t^6 + ct^2}$$

(b) $y'' - 4y' + 3y = 7$

$$r^2 - 4r + 3 = 0$$

$$(r-3)(r-1) = 0$$

$$r = 1, 3$$

$$\Downarrow$$

$$y_1(t) = e^t$$
$$y_2(t) = e^{3t}$$

A particular solution is $y(t) = \frac{7}{3}$.

$$\boxed{y(t) = \frac{7}{3} + c_1 e^t + c_2 e^{3t}}$$

4. (12 points) Consider the nonhomogeneous differential equation

$$y'' + 6y' + 9y = 18e^{-3t}.$$

(a) What is the general solution to the corresponding homogeneous equation (right hand side equal to zero)?

$$r^2 + 6r + 9 = 0$$

$$(r+3)^2 = 0 \longrightarrow r = -3 \quad \left(\text{one real root case}\right)$$

$$y(t) = (c_1 + c_2 t)e^{-3t}$$

(b) Using the method of undetermined coefficients, find a particular solution to the original nonhomogeneous equation.

Since e^{-3t} <u>and</u> te^{-3t} are homogeneous solutions, we should try

$$Y(t) = At^2e^{-3t}$$

$$Y'(t) = A(2t - 3t^2)e^{-3t}$$

$$Y''(t) = A(2 - 12t + 9t^2)e^{-3t}$$

$$\boxed{Y(t) = 9t^2e^{-3t}}$$

$$\Uparrow$$

$$Y''(t) + 6Y'(t) + 9Y(t) = A \cdot 2e^{-3t} = 18e^{-3t} \Rightarrow A = 9$$

(c) Find the solution with $y(0) = 3$ and $y'(0) = -6$.

$$y(t) = e^{-3t}(c_1 + c_2 t + 9t^2)$$

$$y'(t) = e^{-3t}(c_2 + 18t - 3c_1 - 3c_2 t - 27t^2)$$

$$y(0) = c_1 = 3$$

$$y'(0) = c_2 - 3c_1 = -6$$

$$\boxed{\begin{array}{l} c_1 = 3 \\ c_2 = 3 \end{array}}$$

$$\boxed{y(t) = e^{-3t}(3 + 3t + 9t^2)}$$

5. (12 points) Consider the first order differential equation

$$(2x^3 + 5x^2y) + (5x^3 + 2x^2y)\frac{dy}{dx} = 0.$$

$M \uparrow \quad N \uparrow$

(a) Find an integrating factor which makes the above differential equation exact. (Hint: The integrating factor is a function of x.)

$(\mu M)_y = (\mu N)_x$

$\mu'(x) = \frac{M_y - N_x}{N}\mu(x) \longleftarrow \mu M_y = \mu N_x + N\mu_x$

$= \frac{5x^2 - 15x^2 - 4xy}{N}\mu(x)$

$= \frac{-10x^2 - 4xy}{5x^3 + 2x^2y}\mu(x)$

$= -\frac{2}{x}\mu(x) \longrightarrow$ (separable) ... $\mu(x) = x^{-2} \Rightarrow$

$(2x + 5y) + (5x + 2y)\frac{dy}{dx} = 0$ is exact.

$\frac{\partial}{\partial y} \to 5 \leftarrow \frac{\partial}{\partial x}$ ✓

(b) Find the solution to the differential equation with $y(0) = 1$. (An implicitly defined solution is fine.)

$\frac{d}{dx}\left(x^2 + 5xy + y^2\right) = 0 \quad \Rightarrow$

$x^2 + 5xy + y^2 = \text{constant} = 1$

since $y = 1$ when $x = 0$. $\Rightarrow$

$$\boxed{x^2 + 5xy + y^2 = 1}$$

6. (12 points) Given that $y(t) = e^t$ is a solution to

$$(t-1)y'' - ty' + y = 0,$$

find the general two parameter family of solutions to this differential equation.

(these terms cancel)

Try $y(t) = e^t v(t)$.

Then $y'(t) = e^t(v(t) + v'(t))$

$y''(t) = e^t(v(t) + 2v'(t) + v''(t))$ $\Rightarrow$ Plug in and simplify

$$(t-1)e^t(v''+2v') - te^t v' = 0$$

$(t-1)v'' + (t-2)v' = 0$ Let $\underline{w = v'}$. Then

$$(t-1)\frac{dw}{dt} = (2-t)w$$

$$\frac{dw}{w} = \frac{2-t}{t-1}dt = \left(-1 + \frac{1}{t-1}\right)dt$$

$$\ln|w| = -t + \ln|t-1| + C$$

$$v'(t) = w = c_1(t-1)e^{-t}$$

$$v(t) = -c_1 t e^{-t} + c_2$$

$$y(t) = e^t v(t) = \boxed{-c_1 t + c_2 e^{-t}}$$

(Check: note that $y(t) = t$ solves the differential equation.)