Duke University **June 27th, 2013**
Math 353: Ordinary and Partial Differential Equations
Instructor: Hangjun Xu

Final Exam

Name:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

- Do not open this test booklet until you are directed to do so.
- You will have 180 minutes to complete the exam.
- This exam is closed book. But you may use a calculator and a double-sided letter-sized cheat sheet.
- Throughout the exam, show your work so that your reasoning is clear. Otherwise no credit will be given. Circle your answers.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.

Problem	Points	Grade
1	20	
2	20	
3	20	
4	10	
5	25	
6	25	
7	15	
8	20	
9	20	
10	25	
Total	200	

1 (20 points) Solve the initial value problem

$$(2xy^2 + 2y) + (2x^2y + 2x)\frac{dy}{dx} = 0, \quad y(1) = 1.$$

Express the solution y as an explicit function of x in your final answer.

2 (20 points) Find the general solution of the differential equation

$$(x+1)^2 y'' + 4(x+1)y' - 4y = 0$$

for $x > -1$. Show the behavior of the solution as $x \to -1$ in all situations. (*Hint: let* $u = x + 1$.)

3 (20 points) Find the general solution of the differential equation

$$(x-1)y'' - xy' + y = 0$$

for $x > 1$ given that $y_1 = e^x$ is a solution.

4 (10 points) Let $u_{2n}(t)$ where $n = 1, 2, 3, \cdots$ be the unit step functions with base point $2n$. Assume that term-by-term integration of the infinite series is permissible. Find the Laplace transform $F(s)$ of

$$f(t) = 1 + \sum_{n=1}^{\infty} (-1)^n u_{2n}(t).$$

What is the domain of $F(s)$?

5 (25 points) Solve the initial value problem

$$y'' + y = e^t + \delta(t - \pi), \quad y(0) = 0,\ y'(0) = 0.$$

6 (25 points) Find the solution $u(r, \theta)$ of Laplace's equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

outside the circle $r = a$, that satisfies

- $u(a, \theta) = f(\theta)$, for $0 \leq \theta < 2\pi$, where $f(\theta)$ is a 2π-periodic function;
- $u(r, \theta)$ is bounded for $r > a$.

You may use results from Euler equations without rederiving them.

7 (15 points) Let $f(x) = x$ for $0 \leq x < 1$.

(a) (10 points) Find the Fourier sine series of $f(x)$.

(b) (5 points) Sketch the graph of the function to which the series converges on $[-1, 5]$. Label the axes and all important points.

8 (20 points) Use separation of variables to **derive** the solution to the following partial differential equation on the semi-infinite strip $0 < x < a, y > 0$:

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0, y) = 0, & y > 0 \\ u(a, y) = 0, & y > 0 \\ u(x, 0) = f(x), & 0 \leq x \leq a \end{cases}$$

with the additional condition that $\lim_{y \to \infty} u(x, y) = 0$. Justify all steps.

9 (20 points) Use separation of variables to **derive** the solution to the following initial boundary value problem:

$$\begin{cases} u_t + hu = \alpha^2 u_{xx} & \\ u_x(0,t) = 0, & t > 0 \\ u_x(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 \le x \le L \end{cases}$$

where h is a constant. Justify all steps.

10 (25 points) Consider the Sturm-Liouville eigenvalue problem:

$$-[p(x)y']' + q(x)y = \lambda r(x)y,$$

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0, \quad \beta_1 y(1) + \beta_2 y'(1) = 0,$$

where p, q and r are positive and smooth, and $\alpha_2 \neq 0$ and $\beta_2 \neq 0$.

(a) Show that if λ is an eigenvalue and ϕ a corresponding eigenfunction, then

$$\lambda \int_0^1 r\phi^2\, dx = \int_0^1 (p\phi'^2 + q\phi^2)\, dx + \frac{\beta_1}{\beta_2} p(1)\phi^2(1) - \frac{\alpha_1}{\alpha_2} p(0)\phi^2(0).$$

(b) Show that if $q(x) \geq 0$ and if β_1/β_2 and $-\alpha_1/\alpha_2$ are nonnegative, then the eigenvalue λ is nonnegative.

(c) Under the conditions of part (b) show that the eigenvalue λ is strictly positive unless $\alpha_1 = \beta_1 = 0$ and $q(x) = 0$ for each x in $0 \leq x \leq 1$.