

# Midterm, Math 421

## Differential Geometry: Curves and Surfaces in $\mathbb{R}^3$

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February 26, 2013

**Your Name:**

**Honor Pledge Signature:**

**Instructions:** This is a 75 minute, closed book exam. You may bring one  $8\frac{1}{2}'' \times 11''$  piece of paper with anything you like written on it to use during the exam, but nothing else. No collaboration on this exam is allowed. All answers should be written in the space provided, but you may use the backs of pages if necessary.

Express your answers in essay form so that all of your ideas are clearly presented. Partial credit will be given for partial solutions which are understandable. If you want to make a guess, clearly say so. Partial credit will be maximized if you accurately describe what you know and what you are not sure about. Each problem is worth 12 points. Good luck on the exam!

**Problem 1.** Consider the curve in  $\mathbb{R}^3$  parametrized by

$$\alpha(t) = (3 \cos(t), 4 \cos(t), 5 \sin(t)).$$

(a) What is the speed of  $\alpha$ ? Find a *unit speed* reparametrization  $\beta(s)$ .

(b) Using the unit speed reparametrization  $\beta(s)$ , compute the curvature  $\kappa$  of the curve.

(c) Compute all three vectors of the Frenet frame (T, N, B) for the curve  $\beta(s)$ .

(d) Compute the torsion  $\tau$  of the curve  $\beta(s)$ .

(e) Given your answers to parts (b) and (d), what well known shape must this curve trace out?

**Problem 2.** Let  $\beta(s)$  be a unit speed curve in the  $xy$  plane with velocity vector

$$T(s) = \beta'(s) = \cos(\theta(s))e_1 + \sin(\theta(s))e_2$$

where  $e_1 = (1, 0)$  and  $e_2 = (0, 1)$  are the standard basis vectors for the  $xy$  plane.

(a) Prove that  $\theta(s)$  is the angle between  $T(s)$  and  $e_1$ .

(b) Prove that the curvature of the curve is given by

$$\kappa = |\theta'(s)|.$$

(c) Use part (b) to compute the curvature of a circle of radius  $R$ . Use the fact that  $\theta$  changes by  $2\pi$  when you go around the circle once.

**Problem 3.** Consider the surface  $M$  parametrized by

$$x(u, v) = (uv, u^2 - v^2, 3uv - u^2 + v^2).$$

(a) Compute the tangent vectors  $x_u$  and  $x_v$  and the unit normal  $U$ .

(b) Compute  $S(x_u)$  and  $S(x_v)$ . More generally, what is the shape operator  $S(v)$  for any tangent vector  $v$ ?

(c) Given part (b), what shape must the surface  $M$  be? Find an equation for the surface  $M$  in terms of  $x, y, z$ .

**Problem 4.** Let  $k_1$  and  $k_2$  be the principle curvatures of the shape operator at a point  $p$  on a surface  $M$ , with  $k_1 \geq k_2$ . Let  $u$  be any tangent vector to  $M$  at  $p$  of length one.

(a) Define the normal curvature  $k(u)$ .

(b) Prove that  $k(u) \geq k_2$  at the point  $p$  on  $M$ .

(c) Prove that  $k(u) \leq k_1$  at the point  $p$  on  $M$ .

(d) Prove that the average value of  $k(u)$  over the circle of all possible directions equals the mean curvature  $H$  at the point  $p$  on  $M$ .

**Problem 5.** Suppose we have a surface  $M$  parametrized by  $x(u, v)$  with unit normal vector  $U$ . Let  $E, F, G$  and  $l, m, n$  be defined as usual in the book, but suppose that  $F = 0$  everywhere.

(a) Prove that  $\{\frac{x_u}{\sqrt{E}}, \frac{x_v}{\sqrt{G}}, U\}$  forms an orthonormal (length one, mutually perpendicular) basis of vectors at each point on the surface  $M$ .

(b) Prove that

$$x_{uv} = \frac{E_v}{2E}x_u + \frac{G_u}{2G}x_v + mU.$$

**Problem 6.** The Gauss curvature of a surface of revolution of the curve  $\alpha(u) = (g(u), h(u))$  is given by

$$K = \frac{g'(g''h' - h''g')}{h(g'^2 + h'^2)^2}.$$

(a) If we parametrize the curve by arc length so that the velocity of the curve is one, then we may let

$$\alpha'(u) = (g'(u), h'(u)) = (\cos(\theta(u)), \sin(\theta(u))),$$

where  $\theta$  is the angle that the velocity of the curve makes with the  $x$  axis. Compute the Gauss curvature  $K$  in terms of  $h(u)$  and  $\theta(u)$ .

(b) Prove that the only flat ( $K = 0$ ) surfaces of revolution are planes, cones, and cylinders. (Hint: The solution to part (a) makes this easy.)

(c) Let  $M$  be a spherical surface of revolution formed by a curve which begins and ends on the  $x$  axis *vertically*. Using part (a) and the area form formula  $dA = 2\pi h(u)ds$  if you like, prove that

$$\int_M K dA = 4\pi.$$