

Midterm, Math 421

Differential Geometry: Curves and Surfaces in \mathbb{R}^3

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Your Name:

Solutions

Honor Pledge Signature:

Instructions: This is a 75 minute, closed book exam. You may bring one $8\frac{1}{2}'' \times 11''$ piece of paper with anything you like written on it to use during the exam, but nothing else. No collaboration on this exam is allowed. All answers should be written in the space provided, but you may use the backs of pages if necessary.

Express your answers in essay form so that all of your ideas are clearly presented. Partial credit will be given for partial solutions which are understandable. If you want to make a guess, clearly say so. Partial credit will be maximized if you accurately describe what you know and what you are not sure about. Each problem is worth 12 points. Good luck on the exam!

Problem 1. Consider the curve in \mathbb{R}^3 parametrized by

$$\alpha(t) = (3 \cos(t), 4 \cos(t), 5 \sin(t)).$$

(a) What is the speed of α ? Find a *unit speed* reparametrization $\beta(s)$.

$$\alpha'(t) = (-3 \sin t, -4 \sin t, 5 \cos t)$$

$$\text{speed} = |\alpha'(t)| = (9 \sin^2 t + 16 \sin^2 t + 25 \cos^2 t)^{1/2} = \boxed{5}$$

$$s = \int_0^t |\alpha'(t)| dt = \int_0^t 5 dt = 5t \quad ; \quad t = s/5 \Rightarrow$$

$$\beta(s) = \alpha(t) = \boxed{\left(3 \cos \frac{s}{5}, 4 \cos \frac{s}{5}, 5 \sin \frac{s}{5}\right)}$$

(b) Using the unit speed reparametrization $\beta(s)$, compute the curvature κ of the curve.

~~$$\beta(s) = \left(3 \cos \frac{s}{5}, 4 \cos \frac{s}{5}, 5 \sin \frac{s}{5}\right)$$~~
$$\beta'(s) = \left(-\frac{3}{5} \sin\left(\frac{s}{5}\right), -\frac{4}{5} \sin\left(\frac{s}{5}\right), \cos\left(\frac{s}{5}\right)\right)$$

$$\beta''(s) = \left(-\frac{3}{25} \cos\left(\frac{s}{5}\right), -\frac{4}{25} \cos\left(\frac{s}{5}\right), -\frac{1}{5} \sin\left(\frac{s}{5}\right)\right)$$

$$\begin{aligned} \kappa &= |\beta''(s)| = \left(\frac{9}{25^2} \cos^2\left(\frac{s}{5}\right) + \frac{16}{25^2} \cos^2\left(\frac{s}{5}\right) + \frac{1}{25} \sin^2\left(\frac{s}{5}\right)\right)^{1/2} \\ &= \left(\frac{25}{25^2} \cos^2\left(\frac{s}{5}\right) + \frac{1}{25} \sin^2\left(\frac{s}{5}\right)\right)^{1/2} = \left(\frac{1}{25}\right)^{1/2} = \boxed{\frac{1}{5}} \end{aligned}$$

(c) Compute all three vectors of the Frenet frame (T, N, B) for the curve $\beta(s)$.

$$T = \frac{\beta'(s)}{|\beta'(s)|} = \frac{\beta'(s)}{1} = \left(-\frac{3}{5} \sin \frac{s}{5}, -\frac{4}{5} \sin \frac{s}{5}, \cos \frac{s}{5}\right)$$

$$N = \frac{\beta''(s)}{|\beta''(s)|} = \frac{\beta''(s)}{\kappa} = \left(-\frac{3}{5} \cos \frac{s}{5}, -\frac{4}{5} \cos \frac{s}{5}, -\sin \frac{s}{5}\right)$$

$$B = T \times N = \left(\frac{4}{5}, -\frac{3}{5}, 0\right)$$

(d) Compute the torsion τ of the curve $\beta(s)$.

$$\tau = -B'(s) \cdot N = 0$$

(e) Given your answers to parts (b) and (d), what well known shape must this curve trace out?

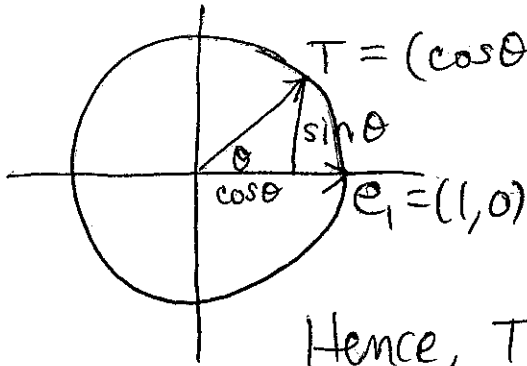
Since the torsion is zero and the curvature is a positive constant, this curve must be a circle (of radius $R = 1/\kappa = 5$).

Problem 2. Let $\beta(s)$ be a unit speed curve in the xy plane with velocity vector

$$T(s) = \beta'(s) = \cos(\theta(s))e_1 + \sin(\theta(s))e_2$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are the standard basis vectors for the xy plane.

(a) Prove that $\theta(s)$ is the angle between $T(s)$ and e_1 .



$T = (\cos \theta, \sin \theta)$ is the unit vector constructed to be an angle θ above the x -axis.

Hence, $T(s) = \cos \theta(s)e_1 + \sin \theta(s)e_2$ makes an angle of size $\theta(s)$ with e_1 .

(b) Prove that the curvature of the curve is given by

$$\kappa = |\theta'(s)|.$$

$$\begin{aligned} \kappa &= |\beta''(s)| = \left| \frac{d}{ds} \beta'(s) \right| = \left| \frac{d}{ds} (\cos \theta(s), \sin \theta(s)) \right| \\ &= \left| (-\theta'(s) \sin \theta(s), \theta'(s) \cos \theta(s)) \right| \quad \text{by the Chain Rule} \\ &= \left(\theta'(s)^2 \sin^2 \theta(s) + \theta'(s)^2 \cos^2 \theta(s) \right)^{1/2} \\ &= \left(\theta'(s)^2 \right)^{1/2} = |\theta'(s)|. \end{aligned}$$

(c) Use part (b) to compute the curvature of a circle of radius R . Use the fact that θ changes by 2π when you go around the circle once.

By symmetry, $\theta'(s)$ is constant on a circle. Also, $\theta(s)$ changes by 2π and s (arc-length) changes by $2\pi R$ when we go around once. Hence,

$$\kappa = |\theta'(s)| = \left| \frac{d\theta}{ds} \right| = \frac{\Delta \theta}{\Delta s} = \frac{2\pi}{2\pi R} = \frac{1}{R}.$$

Problem 3. Consider the surface M parametrized by

$$x(u, v) = (uv, u^2 - v^2, 3uv - u^2 + v^2).$$

(a) Compute the tangent vectors x_u and x_v and the unit normal U .

$$x_u = (v, 2u, 3v - 2u)$$

$$x_v = (u, -2v, 3u + 2v)$$

$$x_u \times x_v = (6u^2 + 4uv + 6v^2 - 4uv, 3uv - 2u^2 - 3uv - 2v^2, -2v^2 - 2u^2)$$

$$= (u^2 + v^2) \cdot (6, -2, -2)$$

$$U = \frac{x_u \times x_v}{|x_u \times x_v|} = \frac{(6, -2, -2)}{(36 + 4 + 4)^{1/2}} = \frac{(6, -2, -2)}{2\sqrt{11}} = \frac{(3, -1, -1)}{\sqrt{11}}$$

(b) Compute $S(x_u)$ and $S(x_v)$. More generally, what is the shape operator $S(v)$ for any tangent vector v ?

$$S(x_u) = \nabla_{x_u} U = U_u = 0$$

$$S(x_v) = \nabla_{x_v} U = U_v = 0$$

$\therefore S(v) = 0$ for all tangent vectors v .

(c) Given part (b), what shape must the surface M be? Find an equation for the surface M in terms of x, y, z .

Since the shape operator is zero, M must be contained in a plane. ~~Therefore, the equation of the plane is~~ The equation of the plane is

$$0 = \vec{U} \cdot ((x, y, z) - (0, 0, 0)) = \frac{3x - y - z}{\sqrt{11}} \Rightarrow$$

$$\boxed{0 = 3x - y - z}$$

Double check:

$$3(uv) - (u^2 - v^2) - (3uv - u^2 + v^2) = 0 \quad \checkmark$$

Problem 4. Let k_1 and k_2 be the principle curvatures of the shape operator at a point p on a surface M , with $k_1 \geq k_2$. Let u be any tangent vector to M at p of length one.

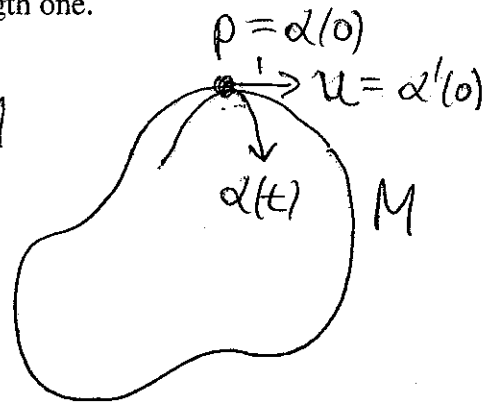
(a) Define the normal curvature $k(u)$.

Let $\alpha(t)$ be any smooth curve in M with $\alpha(0) = p$ and $\alpha'(0) = u$. Then

$$k(u) = \alpha''(0) \cdot U, \text{ where}$$

U is the unit normal to M .

(b) Prove that $k(u) \geq k_2$ at the point p on M .



$k(u) = \alpha''(0) \cdot U = u \cdot S_p(u)$. Let $u = \cos\theta v_1 + \sin\theta v_2$ where v_1 and v_2 are the unit orthogonal eigenvectors of S . Then $k(u) = (\cos\theta v_1 + \sin\theta v_2) \cdot (k_1 \cos\theta v_1 + k_2 \sin\theta v_2) = k_1 \cos^2\theta + k_2 \sin^2\theta = k_1 \cos^2\theta + k_2 (1 - \cos^2\theta)$

(c) Prove that $k(u) \leq k_1$ at the point p on M .

From (b),

$$k(u) = k_1 \cos^2\theta + k_2 \sin^2\theta$$

$$= k_1 (1 - \sin^2\theta) + k_2 \sin^2\theta$$

$$= k_1 - (k_1 - k_2) \sin^2\theta \leq k_1$$

(d) Prove that the average value of $k(u)$ over the circle of all possible directions equals the mean curvature H at the point p on M .

$$\bar{k} = \frac{1}{2\pi} \int_0^{2\pi} k(u) d\theta = \frac{1}{2\pi} \int_0^{2\pi} (k_1 \cos^2\theta + k_2 \sin^2\theta) d\theta$$

Fact: $\int_0^{2\pi} \cos^2\theta d\theta = \pi = \int_0^{2\pi} \sin^2\theta d\theta$. Hence,

$$\bar{k} = \frac{1}{2\pi} (k_1 \cdot \pi + k_2 \cdot \pi) = \frac{k_1 + k_2}{2} = H.$$

Problem 5. Suppose we have a surface M parametrized by $x(u, v)$ with unit normal vector U . Let E, F, G and l, m, n be defined as usual in the book, but suppose that $F = 0$ everywhere.

(a) Prove that $\{\frac{x_u}{\sqrt{E}}, \frac{x_v}{\sqrt{G}}, U\}$ forms an orthonormal (length one, mutually perpendicular) basis of vectors at each point on the surface M .

$$\frac{x_u}{\sqrt{E}} \cdot \frac{x_u}{\sqrt{E}} = \frac{E}{E} = 1 \quad ; \quad \frac{x_v}{\sqrt{G}} \cdot \frac{x_v}{\sqrt{G}} = \frac{G}{G} = 1$$

$x_u, x_v \perp U$ by construction since $U = \frac{x_u \times x_v}{|x_u \times x_v|}$ and $v \cdot (v \times w) = 0$. Finally, $x_u \perp x_v$ since $x_u \cdot x_v = F = 0$.

(b) Prove that

$$x_{uv} = \frac{E_v}{2E} x_u + \frac{G_u}{2G} x_v + mU.$$

Let $x_{uv} = a \frac{x_u}{\sqrt{E}} + b \frac{x_v}{\sqrt{G}} + cU$ in this basis.

Since this is an orthonormal basis,

$$a = x_{uv} \cdot \frac{x_u}{\sqrt{E}} = \frac{1}{2} (x_u \cdot x_u)_v / \sqrt{E} = \frac{1}{2} \frac{E_v}{\sqrt{E}}$$

$$b = x_{uv} \cdot \frac{x_v}{\sqrt{G}} = \frac{1}{2} (x_v \cdot x_v)_u / \sqrt{G} = \frac{1}{2} \frac{G_u}{\sqrt{G}}$$

$$c = x_{uv} \cdot U = m$$

Thus,

$$x_{uv} = \left(\frac{1}{2} \frac{E_v}{\sqrt{E}} \right) \frac{x_u}{\sqrt{E}} + \left(\frac{1}{2} \frac{G_u}{\sqrt{G}} \right) \frac{x_v}{\sqrt{G}} + mU$$

$$= \frac{E_v}{2E} x_u + \frac{G_u}{2G} x_v + mU$$

as desired.

Problem 6. The Gauss curvature of a surface of revolution of the curve $\alpha(u) = (g(u), h(u))$ is given by

$$K = \frac{g'(g''h' - h''g')}{h(g'^2 + h'^2)^2}$$

(a) If we parametrize the curve by arc length so that the velocity of the curve is one, then we may let

$$\alpha'(u) = (g'(u), h'(u)) = (\cos(\theta(u)), \sin(\theta(u))),$$

where θ is the angle that the velocity of the curve makes with the x axis. Compute the Gauss curvature K in terms of $h(u)$ and $\theta(u)$.

$$\begin{array}{l|l} g'(u) = \cos \theta(u) & h'(u) = \sin \theta(u) \\ g''(u) = -\sin \theta(u) \cdot \theta'(u) & h''(u) = \cos \theta(u) \cdot \theta'(u) \end{array}$$

$$K = \frac{\cos \theta (-\sin^2 \theta - \cos^2 \theta) \theta'}{h (\cos^2 \theta + \sin^2 \theta)} = \boxed{-\frac{\theta'(u) \cos \theta(u)}{h(u)}}$$

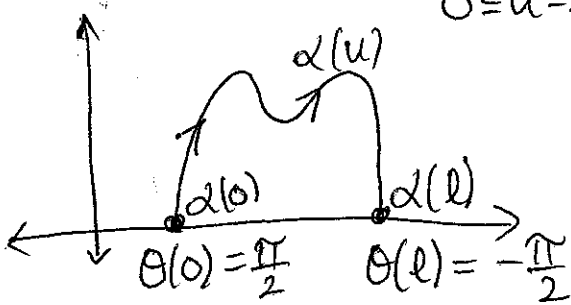
(b) Prove that the only flat ($K = 0$) surfaces of revolution are planes, cones, and cylinders. (Hint: The solution to part (a) makes this easy.)

$$0 = K \Rightarrow \frac{d}{du} (\sin \theta(u)) = 0 \quad (\text{since } h > 0) \\ \Rightarrow \sin \theta(u) \text{ is constant} \Rightarrow \quad (\text{and by Chain Rule})$$

$\theta(u)$ is constant \Rightarrow curve is a line \Rightarrow planes, cones, and cylinders.

(c) Let M be a spherical surface of revolution formed by a curve which begins and ends on the x axis vertically. Using part (a) and the area form formula $dA = 2\pi h(u) ds$ if you like, prove that

l - length of curve $0 \leq u \leq l$ $\int_M K dA = 4\pi$.



$$\begin{aligned} \int_M K dA &= \int_0^l -\frac{\theta'(u) \cos \theta(u)}{h(u)} \cdot 2\pi h(u) ds \\ &= \int_0^l -2\pi \theta'(u) \cos \theta(u) du \\ &= -2\pi \sin \theta(u) \Big|_0^l \\ &= -2\pi \sin(-\frac{\pi}{2}) + 2\pi \sin(\frac{\pi}{2}) = 4\pi! \end{aligned}$$