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2	12
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72

Midterm, Math 421
Differential Geometry: Curves and Surfaces in $\mathbf{R}^3$

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Your Name: Solutions

Honor Pledge Signature:

Instructions: This is a 75 minute, closed book exam. You may bring one $8\frac{1}{2}'' \times 11''$ piece of paper with anything you like written on it to use during the exam, but nothing else. No collaboration on this exam is allowed. All answers should be written in the space provided, but you may use the backs of pages if necessary.

Express your answers in essay form so that all of your ideas are clearly presented. Partial credit will be given for partial solutions which are understandable. If you want to make a guess, clearly say so. Partial credit will be maximized if you accurately describe what you know and what you are not sure about. Each problem is worth 12 points. Good luck on the exam!

Problem 1. Consider the curve in $\mathbf{R}^3$ parametrized by

$$\alpha(t) = (\cos(3t), \sin(3t), 4t).$$

(a) What is the speed of α? Find a *unit speed* reparametrization $\beta(s)$.

$$\alpha'(t) = (-3\sin 3t, 3\cos 3t, 4)$$

$$\text{speed} = |\alpha'(t)| = \sqrt{(3\sin 3t)^2 + (3\cos 3t)^2 + 4^2} = \sqrt{3^2+4^2} = \boxed{5}$$

$$\longrightarrow s = 5t \longrightarrow t = s/5 \longrightarrow$$

$$\beta(s) = \alpha(t) = \alpha(s/5) = \boxed{\left(\cos\frac{3s}{5}, \sin\frac{3s}{5}, \frac{4s}{5}\right)}$$

(b) Using the unit speed reparametrization $\beta(s)$, compute the curvature κ of the curve.

$$\vec{T} = \beta'(s) = \left(-\frac{3}{5}\sin\left(\frac{3s}{5}\right), \frac{3}{5}\cos\left(\frac{3s}{5}\right), \frac{4}{5}\right)$$

$$\kappa\vec{N} = \beta''(s) = \left(-\frac{9}{25}\cos\left(\frac{3s}{5}\right), -\frac{9}{25}\sin\left(\frac{3s}{5}\right), 0\right)$$

$$\kappa = |\beta''(s)| = \sqrt{\left(\frac{9}{25}\cos\frac{3s}{5}\right)^2 + \left(\frac{9}{25}\sin\left(\frac{3s}{5}\right)\right)^2 + 0^2} = \boxed{\frac{9}{25}}$$

(c) Compute all three vectors of the Frenet frame (T, N, B) for the curve $\beta(s)$.

From (b),

$$\vec{T} = \left(-\frac{3}{5}\sin\left(\frac{3s}{5}\right), \frac{3}{5}\cos\left(\frac{3s}{5}\right), \frac{4}{5}\right)$$

$$\vec{N} = \left(-\cos\left(\frac{3s}{5}\right), -\sin\left(\frac{3s}{5}\right), 0\right)$$

$$\vec{B} = \vec{T}\times\vec{N}$$

$$= \left(\frac{4}{5}\sin\left(\frac{3s}{5}\right), -\frac{4}{5}\cos\left(\frac{3s}{5}\right), \frac{3}{5}\right)$$

(d) Compute the torsion τ of the curve $\beta(s)$.

$$\tau = \vec{N}'(s)\cdot\vec{B} \qquad \left(\text{or use } \tau = -\vec{B}'(s)\cdot\vec{N}\right)$$

$$= \left(\frac{3}{5}\sin\left(\frac{3s}{5}\right), -\frac{3}{5}\cos\left(\frac{3s}{5}\right), 0\right)\cdot\vec{B}$$

$$= \boxed{\frac{12}{25}}$$

Problem 2. Suppose a unit speed curve $\alpha(s)$ has constant curvature $\kappa > 0$ and zero torsion τ.
(a) Show that

$$\gamma(s) = \alpha(s) + \frac{1}{\kappa}N$$

is a constant curve; that is, show that $\gamma(s) = p$ for some fixed point p.

$$\gamma'(s) = \alpha'(s) + \frac{1}{\kappa}N'(s)$$
$$= T + \frac{1}{\kappa}(-\kappa T + \tau B) = 0$$

Hence, $\gamma(s) = p$ for some fixed point p.

(b) Using part (a), prove that the curve $\alpha(s)$ is part of a circle centered at the point p. What is the radius of the circle?

$$\alpha(s) + \frac{1}{\kappa}N = p$$

$$\alpha(s) - p = -\frac{1}{\kappa}N$$

$$(*) \quad |\alpha(s) - p| = \frac{1}{\kappa}$$

Since $\tau = 0$, $\alpha(s)$ is in a plane. $N = \frac{\alpha''(s)}{|\alpha''(s)|}$ is in the same plane, as is p. Thus, $(*)$ proves that $\alpha(s)$ is part of a circle (in this plane) of radius $1/\kappa$.

Problem 3.

(a) Given a surface M with unit normal vector field U, define the shape operator $S_p(v)$ at the point p on M, where v is a tangent vector to M at p.

$$S_p(\vec{v}) = -\nabla_{\vec{v}} \vec{U} \Big|_p$$

(b) Prove that $S_p(v)$ is also a tangent vector to M at p. (Hence, $S_p : T_pM \to T_pM$.)

$$1 = \vec{U} \cdot \vec{U}$$

differentiate in $\vec{v}$ direction

$$0 = 2\,\vec{U} \cdot (\nabla_v \vec{U})$$

$$\vec{U} \perp S_p(\vec{v})$$

$S_p(\vec{v})$ is a tangent vector to M at p.

(c) Prove that S_p is symmetric as follows: Given a coordinate chart $\vec{x}(u, v)$, show that

$$S_p(\vec{x}_u) \cdot \vec{x}_v = \vec{x}_u \cdot S_p(\vec{x}_v).$$

$$0 = \vec{U} \cdot \vec{X}_u$$

$$0 = \vec{X}_v[\vec{U} \cdot \vec{X}_u] = \vec{U} \cdot \vec{X}_{uv} - S_p(\vec{X}_v) \cdot \vec{X}_u$$

Thus,

$$\boxed{S_p(\vec{X}_v) \cdot \vec{X}_u} = \vec{U} \cdot \vec{X}_{uv} = \vec{U} \cdot \vec{X}_{vu} = \boxed{S_p(\vec{X}_u) \cdot \vec{X}_v}.$$

Problem 4. Suppose M is parametrized by $\vec{x}(u,v)$ with unit normal vector field U and consists entirely of umbilic points. That is, suppose

$$S_p(\vec{v}) = k(p)\,\vec{v},$$

where $k(p)$ is a real-valued function for $p \in M$.

(a) Prove that $k(p)$ equals a constant (call it k_0) on M.

$$-U_{uv} = -(U_u)_v = (S_p(\vec{X}_u))_v = (k\vec{X}_u)_v = k\vec{X}_{uv} + k_v\vec{X}_u$$

$$\parallel$$

$$-U_{vu} = -(U_v)_u = (S_p(X_v))_u = (k\vec{X}_v)_u = k\vec{X}_{uv} + k_u\vec{X}_v.$$

Thus, $k_v\vec{X}_u = k_u\vec{X}_v \Rightarrow k_u = 0 = k_v$ since $\vec{X}_u, \vec{X}_v$ are linearly independent.

Hence $k(p) = k_0$, a <u>constant</u>.

(b) If $k_0 = 0$, show that M is contained in a plane. (From scratch - do not quote the theorem that zero second fundamental form implies that M is in a plane - prove this.)

Let $\alpha(t)$ be a curve on M from any point $p \in M$ to a fixed point $q \in M$, where $\alpha(0) = p$ and $\alpha(1) = q$.

$$\frac{d}{dt}\left\{(\alpha(t) - \alpha(0)) \cdot \vec{U}(\alpha(t))\right\}$$

$$= \alpha'(t) \cdot \vec{U}(\alpha(t)) + (\alpha(t) - \alpha(0)) \cdot (-S_{\alpha(t)}(\alpha'(t)))$$

$$= 0 + 0 = 0.$$

Hence, $\{\ \} =$ constant. Since $\{\} = 0$ when $t = 0$, $\{(\alpha(t) - \alpha(0)) \cdot \vec{U}(\alpha(t))\} = 0$ for all t. Plug in $t = 1$ to get

$\boxed{(q-p) \cdot \vec{U}(q) = 0}$ for all $p \in M$. But this is the equation of a plane through q with normal $\vec{U}(q)$. Hence, M is contained in a plane.

(c) If $k_0 \neq 0$, then show that

$$\vec{y}(u,v) = \vec{x}(u,v) + \frac{1}{k_0}U$$

is a constant. That is, show that $\vec{y}(u,v) = p$ for some fixed point p.

$$\frac{\partial}{\partial u}(\vec{y}(u,v)) = \vec{y}_u = \vec{X}_u + \frac{1}{k_0}(-S(\vec{X}_u)) = \vec{X}_u + \frac{1}{k_0}(-k_0\vec{X}_u) = 0$$

$$\frac{\partial}{\partial v}(\vec{y}(u,v)) = \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots = 0.$$

Hence, $\boxed{\vec{y}(u,v) = \vec{p}}$ for some fixed point p.

(d) Using part (c), prove that the surface is part of a sphere centered at the point p. What is the radius of the sphere?

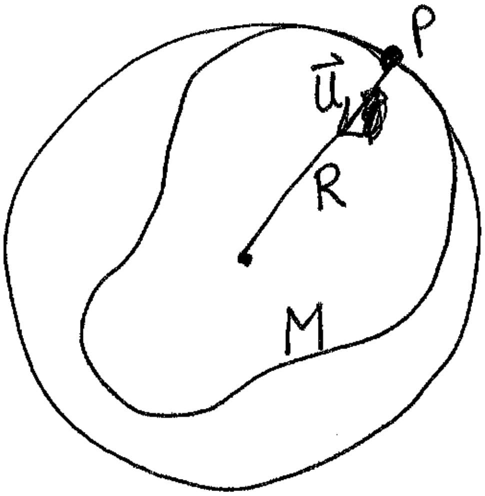
Thus, $\vec{X}(u,v) + \frac{1}{k_0}\vec{U} = \vec{p}$

$$\vec{X}(u,v) - \vec{p} = -\frac{1}{k_0}\vec{U}$$

$$|\vec{X}(u,v) - \vec{p}| = \frac{1}{k_0}.$$

Thus, the surface is part of a sphere centered at $\vec{p}$ with radius $1/k_0$.

Problem 5. (a) Prove that on every compact surface (surface without boundary contained in a large ball) $M \subset \mathbf{R}^3$, there is at least one point of M with positive Gauss curvature K.



Suppose smallest ~~ball~~ sphere enclosing M is tangent to M at p. Let $\vec{u}$ point inwards. Then at p,

$$k_1, k_2 \geq \frac{1}{R} \Rightarrow$$

$$K = k_1 k_2 \geq \frac{1}{R^2} > 0.$$

(b) Prove that there does not exist a compact minimal surface in $\mathbf{R}^3$.

Minimal $\Leftrightarrow 0 = H = \frac{k_1 + k_2}{2}$ at each point $p \in M$

$\Downarrow$

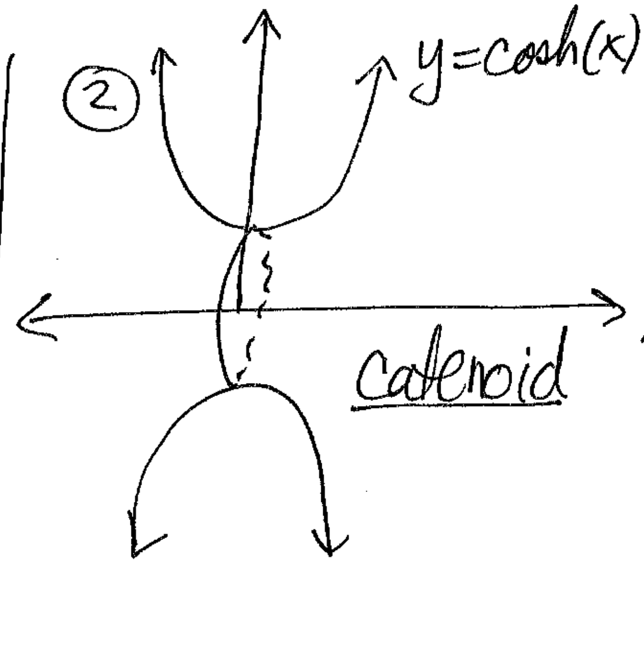
$$\boxed{K = k_1 k_2 = -k_1^2 \leq 0 \text{ everywhere on } M.} \Leftarrow k_1 = -k_2$$

But this violates our theorem in (a).

(c) Give two examples of a minimal surface in $\mathbf{R}^3$.

① $\boxed{\text{any plane}}$

②



catenoid

③ Helicoid

$$\vec{X}(u,v) = \begin{pmatrix} u\cos v \\ u \sin v \\ v \end{pmatrix}$$

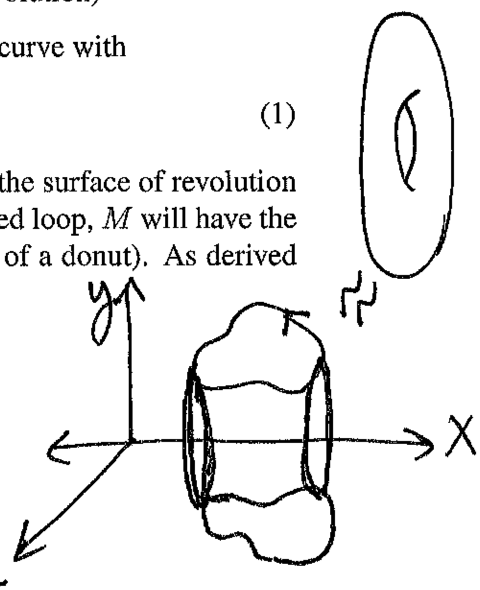
Problem 6. (The Gauss Bonnet Theorem for a Toroidal Surface of Revolution)

Let $\alpha(s) = (x(s), y(s))$, $a \leq s \leq b$, be a *unit speed* smooth closed curve with

$$\alpha(a) = \alpha(b) \quad \text{and} \quad \alpha'(a) = \alpha'(b) \tag{1}$$

in the xy plane which does not intersect itself or the x-axis. Let M be the surface of revolution created by rotating the curve α around the x-axis. Since α forms a closed loop, M will have the topology of a torus (that is, be topologically equivalent to the surface of a donut). As derived on page 119 of the book, the formula for the Gauss curvature of M is

$$K = \frac{x'(s)}{y(s)} \left(x''(s)y'(s) - y''(s)x'(s) \right).$$



Using the above formula, prove that

$$\int_M K dA = c$$

for any such closed curve α, for some constant c. Compute the constant c (which is a familiar number).

Hints: You will need to use $dA = 2\pi y\, ds$, where ds is the length form of the curve, and the fact that $x'(s)^2 + y'(s)^2 = 1$.

$$\frac{d}{ds} \hookrightarrow 2x'x'' + 2y'y'' = 0 \rightarrow x'x'' = -y'y''$$

$$K = \frac{1}{y}\left(x'x'' \cdot y' - y''(x')^2\right) = \frac{1}{y}\left(-y'' \cdot (y')^2 - y''(x')^2\right)$$

$$= -\frac{y''}{y}\left[(x')^2 + (y')^2\right] = \boxed{-\frac{y''(s)}{y(s)}}.$$

$$\int_M K dA = \int_a^b -\frac{y''(s)}{y(s)} \cdot 2\pi y\, ds = \int_a^b -2\pi y''(s) ds$$

$$= -2\pi y'(s)\Big]_a^b = -2\pi\left(y'(b) - y'(a)\right) = \boxed{0.}$$
