# Midterm Exam, Math 421
# Differential Geometry: Curves and Surfaces in $\mathbf{R}^3$

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**Your Name:** Solutions

**Honor Pledge Signature:**

**Instructions:** This is a 75 minute, closed book exam. You may bring one $8\frac{1}{2}'' \times 11''$ piece of paper with anything you like written on it to use during the exam, but nothing else. No collaboration on this exam is allowed. All answers should be written in the space provided, but you may use the backs of pages if necessary.

Express your answers in essay form so that all of your ideas are clearly presented. Partial credit will be given for partial solutions which are understandable. If you want to make a guess, clearly say so. Partial credit will be maximized if you accurately describe what you know and what you are not sure about. Each problem is worth 12 points. Good luck on the exam!

| Question | Points | Score |
|---|---|---|
| 1 | 12 | |
| 2 | 12 | |
| 3 | 12 | |
| 4 | 12 | |
| 5 | 12 | |
| 6 | 12 | |
| Total | 72 | |

**Problem 1.** Consider the curve in $\mathbf{R}^3$ parametrized by

$$\alpha(t) = (5\cos(2t), 3\sin(2t), 4\sin(2t)).$$

(a) What is the speed of $\alpha$? Find a *unit speed* reparametrization $\beta(s)$.

$$\alpha'(t) = (-10\sin(2t), 6\cos(2t), 8\cos(2t))$$

$$\text{speed} = \frac{ds}{dt} = |\alpha'(t)| = \sqrt{100\sin^2(2t) + (36+64)\cos^2(2t)}$$

$$= \sqrt{100}$$

$$= 10$$

Hence, $s = 10t \longrightarrow t = s/10$

$$\beta(s) = \left(5\cos\frac{s}{5}, 3\sin\frac{s}{5}, 4\sin\frac{s}{5}\right)$$

(b) Using the unit speed reparametrization $\beta(s)$, compute the curvature $\kappa$ of the curve.

$$\vec{T} = \beta'(s) = \left(-\sin\frac{s}{5}, \frac{3}{5}\cos\frac{s}{5}, \frac{4}{5}\cos\frac{s}{5}\right)$$

$$\kappa\vec{N} = \beta''(s) = \left(-\frac{1}{5}\cos\frac{s}{5}, -\frac{3}{25}\sin\frac{s}{5}, -\frac{4}{25}\sin\frac{s}{5}\right)$$

$$\kappa = \|\beta''(s)\| = \sqrt{\frac{1}{25}\cos^2\frac{s}{5} + \frac{9+16}{25^2}\sin^2\frac{s}{5}}$$

$$= \sqrt{\frac{1}{25}}$$

$$= \frac{1}{5}$$

(c) Compute all three vectors of the Frenet frame (T, N, B) for the curve $\beta(s)$.

$$\left.\begin{aligned}\vec{T} &= \left(-\sin\frac{s}{5},\ \frac{3}{5}\cos\frac{s}{5},\ \frac{4}{5}\cos\frac{s}{5}\right)\\ \vec{N} &= \left(-\cos\frac{s}{5},\ -\frac{3}{5}\sin\frac{s}{5},\ -\frac{4}{5}\sin\frac{s}{5}\right)\end{aligned}\right\}\text{ see (b)}$$

$$\vec{B} = \vec{T}\times\vec{N}$$
$$= \left(0,\ -\frac{4}{5},\ \frac{3}{5}\right)$$

(d) Compute the torsion $\tau$ of the curve $\beta(s)$. Based on this and your answer to part (b), what is this curve?

$$\tau = -\vec{B}'(s)\cdot\vec{N} = 0.$$

Since $\tau = 0$, this is a planar curve (contained in a plane).

Since $\kappa = \text{constant} = \frac{1}{5}$, this is a $\boxed{\text{circle}}$ of radius 5.

**Problem 2.**

(a) Given a surface $M$ with unit normal vector field $U$, define the shape operator $S_p(v)$ at the point $p$ on $M$, where $v$ is a tangent vector to $M$ at p.

$$S_p(v) = -\nabla_v U$$

(b) Prove that $S_p(v)$ is also a tangent vector to $M$ at $p$. (Hence, $S_p : T_pM \to T_pM$.)

$$0 = v[1] = v[U \cdot U] = 2(\nabla_v U) \cdot U$$

Hence, $\nabla_v U \perp U$, so $S_p(v) \in T_pM$.

(c) Prove that $S_p$ is symmetric as follows: Given a coordinate chart $\vec{x}(u, v)$, show that

$$S_p(\vec{x}_u) \cdot \vec{x}_v = \vec{x}_u \cdot S_p(\vec{x}_v).$$

$$0 = \vec{X}_u[0] = \vec{X}_u[\vec{X}_v \cdot \vec{U}] = \vec{X}_{uv} \cdot \vec{U} + (\nabla_{\vec{X}_u} \vec{U}) \cdot \vec{X}_v$$

Hence,

$$S_p(\vec{X}_u) \cdot \vec{X}_v = \vec{X}_{uv} \cdot \vec{U}$$

Similarly, $S_p(\vec{X}_v) \cdot \vec{X}_u = \vec{X}_{vu} \cdot \vec{U}$

equal since $\vec{X}_{uv} = \vec{X}_{vu}$

Thus, $S_p(\vec{X}_u) \cdot \vec{X}_v = \vec{X}_u \cdot S_p(\vec{X}_v)$

**Problem 3.** Suppose $M$ is parametrized by $\vec{x}(u, v)$ with unit normal vector field $U$ and consists entirely of umbilic points. That is, suppose

$$S_p(\vec{v}) = k(p)\ \vec{v},$$

where $k(p)$ is a real-valued function for $p \in M$.

(a) Prove that $k(p)$ equals a constant (call it $k_0$) on $M$.

$$U_{uv} = (U_u)_v = (-S_p(\vec{X}_u))_v = (-k\vec{X}_u)_v = -k_v X_u - k\vec{X}_{uv}$$

$$\| $$

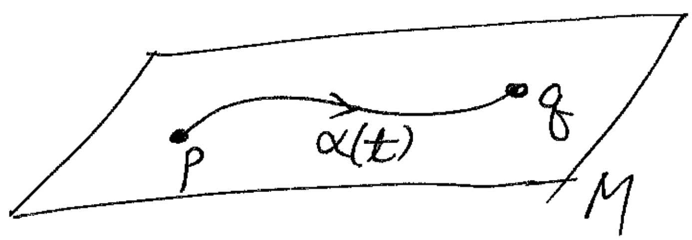
$$U_{vu} = (U_v)_u = (-S_p(X_v))_u = (-kX_v)_u = -k_u X_v - k\vec{X}_{vu}$$

Since $\vec{X}_{uv} = \vec{X}_{vu}$, $\Rightarrow$ $k_u \vec{X}_v = k_v \vec{X}_u \Rightarrow k_u = 0 = k_v$.

since $\vec{X}_u$ and $\vec{X}_v$ are linearly independent.

(b) If $k_0 = 0$, show that $M$ is contained in a plane. (From scratch - do not quote the theorem that zero second fundamental form implies that $M$ is in a plane - prove this.)

Let $\alpha(t)$ be a curve on $M$ from any point $p$ to a fixed point $q$, where $\alpha(0) = p$ and $\alpha(1) = q$.




$$\frac{d}{dt}\left\{(\alpha(t) - \alpha(0)) \cdot \vec{U}(\alpha(t))\right\} = \underset{0}{\alpha'(t) \cdot \vec{U}(\alpha(t))} + (\alpha(t) - \alpha(0)) \cdot \underset{0}{(-S(\alpha'(t)))}$$

$$= 0 + 0 = 0.$$

Hence, $\{\ \} = \text{const} = 0$ since it is zero when $t = 0$. Thus, plugging in $t = 1$, we get

$(q - p) \cdot \vec{U}(q) = 0$, $\forall p$, which is the equation of a plane through $q$ with normal $\vec{U}(q)$.

(c) If $k_0 \neq 0$, then show that

$$\vec{y}(u,v) = \vec{x}(u,v) + \frac{1}{k_0}U$$

is a constant. That is, show that $\vec{y}(u,v) = p$ for some fixed point $p$.

$$\vec{y}_u = \vec{X}_u + \frac{1}{k_0}(\nabla_{\vec{X}_u} U) = \vec{X}_u + \frac{1}{k_0}(-k_0 \vec{X}_u) = \vec{0}$$

$$\vec{y}_v = \vec{X}_v + \frac{1}{k_0}(\nabla_{\vec{X}_v} U) = \vec{X}_v + \frac{1}{k_0}(-k_0 \vec{X}_v) = \vec{0}.$$

Hence, $\vec{y}(u,v) = \vec{p}$, for some fixed point $\vec{p}$.

(d) Using part (c), prove that the surface is part of a sphere centered at the point $p$. What is the radius of the sphere?

$$\vec{X}(u,v) + \frac{1}{k_0}\vec{U} = \vec{p}$$

$$\vec{X}(u,v) - \vec{p} = -\frac{1}{k_0}\vec{U}$$

$$|\vec{X}(u,v) - \vec{p}| = \frac{1}{|k_0|} = R$$

Hence, $\vec{X}(u,v)$ is on the sphere of radius $R = \frac{1}{|k_0|}$ centered at $\vec{p}$.

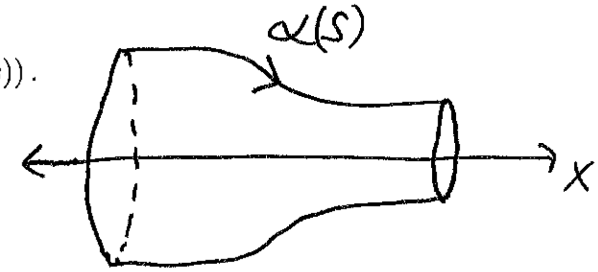
**Problem 4.** (The Gauss Bonnet Theorem for a Cylinder-like Surface of Revolution)

Let $\alpha(s) = (x(s), y(s))$, $a \le s \le b$, be a *unit speed* smooth [redacted] curve with

$$\alpha'(a) = \alpha'(b) = (1,0) \quad \longrightarrow y'(a) = y'(b) = 0 \qquad (1)$$

in the $xy$ plane which does not intersect itself or the $x$-axis. Let $M$ be the surface of revolution created by rotating the curve $\alpha$ around the $x$-axis, which topologically will be like a cylinder. As derived on page 119 of the book, the formula for the Gauss curvature of $M$ is

$$K = \frac{x'(s)}{y(s)}\left(x''(s)y'(s) - y''(s)x'(s)\right).$$




Using the above formula, prove that

$$\int_M K\,dA = c$$

for any such curve $\alpha$, for some constant c. Compute the constant c (which is a familiar number).

**Hints:** You will need to use $dA = 2\pi y\,ds$, where $ds$ is the length form of the curve, and the fact that $x'(s)^2 + y'(s)^2 = 1$.

$$\downarrow$$

$$2x'x'' + 2y'y'' = 0 \longrightarrow x'x'' = -y'y''.$$

$$\int_M K\,dA = \int_a^b \frac{x'}{y}(x''y' - y''x') \cdot 2\pi y\,ds$$

$$= \int_a^b 2\pi\,(\underline{\underline{x'x''}}y' - x'y''x')\,ds$$

$$= \int_a^b 2\pi\,(-y'y''y' - x'y''x')\,ds$$

$$= \int_a^b -2\pi \cdot y''((x')^2 + (y')^2)\,ds$$

$$= \int_a^b -2\pi\, y''(s)\,ds$$

$$= -2\pi\, y'(s)\Big]_a^b = \boxed{0}$$

**Problem 5.** Consider the surface $M$ parametrized by

$$\vec{x}(u,v) = (1+u+v, 3+u-2v, \sqrt{25+2uv-2u^2-5v^2}),$$

for $-1 < u < 1$ and $-1 < v < 1$.

(a) Compute $\vec{x}_u$, $\vec{x}_v$, and $U$.

$$\vec{X}_u = \left(1,\ 1,\ \left[25+2uv-2u^2-5v^2\right]^{-1/2}(v-2u)\right)$$

$$\vec{X}_v = \left(1,\ -2,\ \left[25+2uv-2u^2-5v^2\right]^{-1/2}(u-5v)\right)$$

$$\vec{X}_u \times \vec{X}_v = \left(\left[\ \right]^{-1/2}(-3u-3v),\ \left[\ \right]^{-1/2}(-3u+6v),\ -3\right)$$

$$= -3\left[\ \right]^{-1/2}\underbrace{\left(u+v,\ u-2v,\ \left[\ \right]^{1/2}\right)}_{\text{Check: This vector has length 5!}}$$

$$\vec{U} = \frac{\vec{X}_u \times \vec{X}_v}{|\vec{X}_u \times \vec{X}_v|} = -\frac{1}{5}\left(u+v,\ u-2v,\ \left[25+2uv-2u^2-5v^2\right]^{1/2}\right)$$

(b) Compute $S_p(\vec{x}_u)$ and $S_p(\vec{x}_v)$.

$$S_p(X_u) = -\nabla_{X_u} U = -U_u$$
$$= \frac{1}{5}\left(1, 1, [\ ]^{-1/2}(v-2u)\right) = \frac{1}{5} \vec{X}_u.$$

$$S_p(\vec{X}_v) = -\nabla_{X_v} U = -U_v$$
$$= \frac{1}{5}\left(1, -2, [\ ]^{-1/2}(u-5v)\right) = \frac{1}{5} \vec{X}_v.$$

(c) What is the shape operator $S_p(\vec{v})$, for any tangent vector $\vec{v}$?

$$S_p(\vec{v}) = S_p(a\vec{X}_u + b\vec{X}_v) = a \cdot \frac{1}{5}\vec{X}_u + b \cdot \frac{1}{5}\vec{X}_v$$
$$= \frac{1}{5}(a\vec{X}_u + b\vec{X}_v) = \frac{1}{5}\vec{V}.$$

(d) What is the surface $M$? Describe it as well as you can.

Since $k_1 = k_2 = \frac{1}{5}$ everywhere, this surface is contained in a sphere of radius 5.

(The center of the sphere is $(1, 3, 0)$.)

Note: $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\cos\theta\sin\theta = (\sin 2\theta)/2$

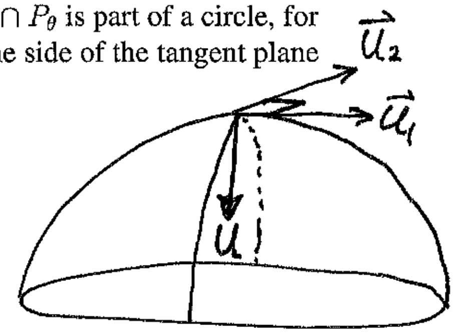
**Problem 6. Challenge Problem!** Let $U$ be the unit normal to the **smooth** surface $M$ at the point $p \in M$, and let $u_1$ and $u_2$ be an orthonormal basis for the tangent plane $T_pM$ to $M$ at $p$. Define

$$u(\theta) = \cos(\theta)u_1 + \sin(\theta)u_2,$$

and let $P_\theta$ be the plane through $p$ containing the vectors $U$ and $u(\theta)$.

Now suppose that $p$ and $M$ have the interesting property that $M \cap P_\theta$ is part of a circle, for all planes $P_\theta$ through the point $p$, with all of these circles on the same side of the tangent plane $T_pM$ to $M$ at $p$. Furthermore, suppose that

- $M \cap P_0$ is part of a circle of radius 3,
- $M \cap P_{\pi/4}$ is part of a circle of radius 2, and
- $M \cap P_{\pi/2}$ is part of a circle of radius 3.




What is the radius of the circle containing $M \cap P_{3\pi/4}$?

Choose $\vec{U}$ to point towards the ~~circular~~ circular cross sections. Then

$$k(\theta) = k(\vec{u}(\theta)) = \vec{u}\cdot S(\vec{u}) = \vec{U}\cdot\beta''(0) = \kappa = \frac{1}{R}$$

where $R$ is the radius of $M\cap P_\theta$. Furthermore,

$$k(\theta) = (\cos\theta\,\vec{u}_1 + \sin\theta\,\vec{u}_2)\cdot S(\cos\theta\,\vec{u}_1 + \sin\theta\,\vec{u}_2)$$

$$= \cos^2\theta\,(\vec{u}_1\cdot S(\vec{u}_1)) + \sin^2\theta\,(u_2\cdot S(u_2)) + \cos\theta\sin\theta\,(u_1\cdot S(u_2) + u_2\cdot S(u_1))$$

$$= A\cos^2\theta + B\sin^2\theta + C\sin\theta\cdot\cos\theta \quad \text{for some } A, B, C$$

$$= a + b\cos 2\theta + c\sin 2\theta \quad \text{for some } a, b, c$$

$\frac{1}{3} = k(0) = a+b$

$\frac{1}{2} = k(\frac{\pi}{4}) = a+c$

$\frac{1}{3} = k(\frac{\pi}{2}) = a-b$

$\rightarrow$ $a = \frac{1}{3}$, $b = 0$, $c = \frac{1}{6}$

$\rightarrow$ $k(\frac{3\pi}{4}) = a - c = \frac{1}{3} - \frac{1}{6} = \frac{1}{6} \rightarrow$ $\boxed{R = 6}$.