Final Exam, Math 421
Differential Geometry: Curves and Surfaces in $\mathbf{R}^3$

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Thursday, May 6, 2017

Your Name:

Instructions: This is a 3 hour, closed book exam. All answers should be written in the space provided, but you may use the backs of pages if necessary.

Express your answers in essay form so that all of your ideas are clearly presented. Partial credit will be given for partial solutions which are understandable. If you want to make a guess, clearly say so. Partial credit will be maximized if you accurately describe what you know and what you are not sure about. Each problem is worth 12 points. Good luck on the exam!

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
7	12	
8	12	
9	12	
10	12	
11	12	
12	12	
Total	144	

Problem 1. Consider the curve parametrized by

$$\alpha(t) = (\cos(e^t), \sin(e^t), e^t).$$

(a) What is the speed of $\alpha(t)$?

$$v = |\alpha'(t)| = |e^t(-\sin(e^t), \cos(e^t), 1)| = \sqrt{2}e^t = \frac{ds}{dt}$$

(b) Find a *unit speed* reparametrization $\beta(s)$ of this curve.

$$ds = \sqrt{2}e^t dt$$

$$s = \sqrt{2}e^t$$

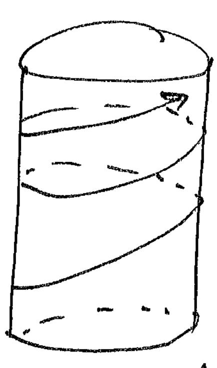
$$\beta(s) = \alpha(t) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$$

(c) What is the curvature of $\beta(s)$?

$$\kappa = |\beta''(s)| = \left|\left(-\frac{1}{2}\cos\frac{s}{\sqrt{2}}, -\frac{1}{2}\sin\frac{s}{\sqrt{2}}, 0\right)\right| = \frac{1}{2}$$

(d) Define a surface on which $\beta(s)$ is a geodesic.

$$x^2 + y^2 = 1$$



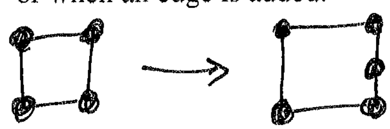
cylinder

Problem 2.

(a) Define the Euler characteristic of a surface.

$$\chi = F - E + V$$

(b) Prove that the Euler characteristic of a triangulation does not change when a vertex is added, or when an edge is added.



$E \to E+1$

$V \to V+1$

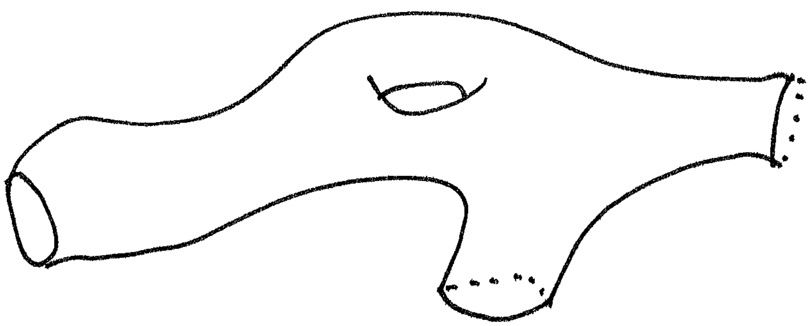
χ is unchanged.

$E \to E+1$

$F \to F+1$

χ is unchanged.

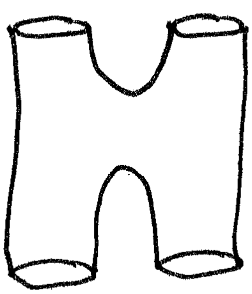
(c) What is the Euler characteristic of this surface?



$\chi($  $) = 0$, so

$$\chi = 0 - 1 - 1 - 1 = -3$$

(d) What is the average value of the Gauss curvature of this surface of total area 10? You may assume that the boundary curves are geodesics.



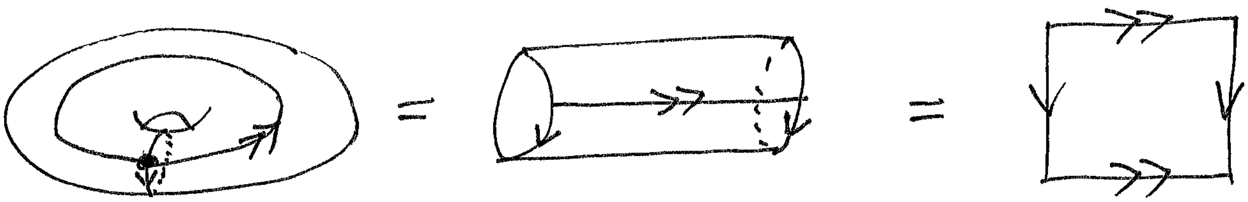
$\chi($  $) = 2$, so

$$\chi = 2 - 1 - 1 - 1 - 1 = -2$$

$$\bar{K} = \frac{1}{10}\int K\,dA = \frac{1}{10}\cdot(2\pi\chi) = -\frac{4\pi}{10} = -\frac{2\pi}{5}$$

Problem 3.

(a) Compute the Euler characteristic of a torus (surface of a donut).



$V = 1$
$E = 2$
$F = 1$

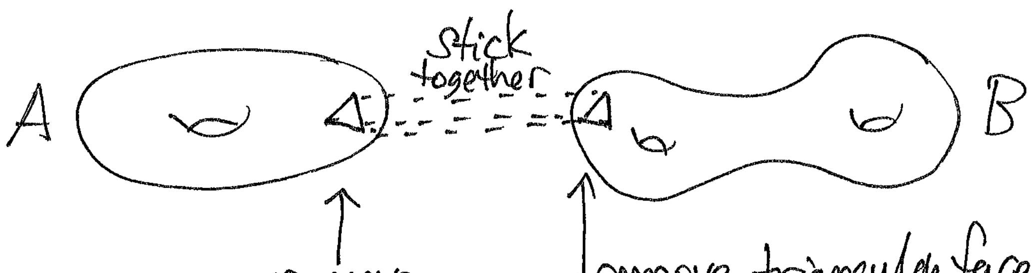
$\chi = 0.$

$$\chi = 1 - 2 - 1 = 0.$$

(b) Prove the following connect sum identity for Euler characteristics,

$$\chi(A\#B) = \chi(A) + \chi(B) - 2$$

where A and B are compact surfaces with boundary.

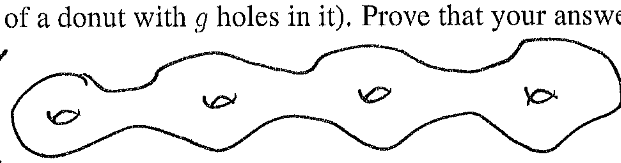


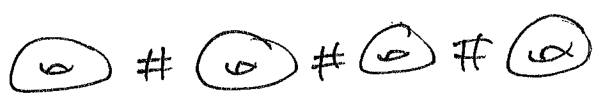
remove triangular face

remove triangular face, 3 vertices, and 3 edges.

$$\chi(A\#B) = \chi(A) + \chi(B) - 1 - 1 + 3 - 3 = \chi(A) + \chi(B) - 2$$

(c) Use parts (a) and (b) to compute the Euler characteristic of a surface of genus g (the surface of a donut with g holes in it). Prove that your answer is correct.

$\chi($  $)$ (g holes)

$= \chi($  $)$ (g tori)

$= 0 - 2 - 2 - 2$ (g-1 connect sums)

$= -2(g-1) = 2 - 2g$

Problem 4. Suppose you are a two dimensional being living on a surface without boundary in an alternate universe. Though the surface is quite lumpy and not at all symmetrical, your society uses a system of measurements so that the total area of the surface on which you all live is 10.

(a) Through very careful measurement, you determine that the average value of the Gauss curvature of the surface is between 1.0 and 1.5. What is the topological name of the surface on which you live? (S^2, RP^2, T^2, K^2, $T^2 \# T^2$, ...)

$$10 \leq \int K\,dA = 2\pi \chi \leq 15$$

$$\frac{10}{2\pi} \leq \chi \leq \frac{15}{2\pi} \Rightarrow \chi = 2 \text{ since } \chi \text{ must be an integer.}$$

Hence, the surface is a topological sphere (S^2).

(b) Suppose instead that the average value of the Gauss curvature of the surface is between 0.5 and 0.75. Now what is the topological name of the surface on which you live? (S^2, RP^2, T^2, K^2, $T^2 \# T^2$, ...)

$$5 \leq \int K\,dA = 2\pi \chi \leq 7.5$$

$$\frac{5}{2\pi} \leq \chi \leq \frac{7.5}{2\pi} \Rightarrow \chi = 1 \Rightarrow$$

$\mathbb{RP}^2$

(c) Suppose instead that the average value of the Gauss curvature of the surface is between -0.5 and -0.75. Now what is the topological name of the surface on which you live? (S^2, RP^2, T^2, K^2, $T^2 \# T^2$, ...)

$$-7.5 \leq \int K\,dA = 2\pi \chi \leq -5$$

$$-\frac{7.5}{2\pi} \leq \chi \leq -\frac{5}{2\pi} \Rightarrow \chi = -1$$

$\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$ $\quad (= K^2 \# \mathbb{RP}^2)$

Problem 5. Define a geodesic of a surface M in R^3 to be any curve $\alpha(t)$ on M such that $\alpha''(t)$ is perpendicular to M.

(a) Prove that a geodesic has constant speed.

$$\frac{d}{dt} v^2 = \frac{d}{dt}\left(\alpha'(t)\cdot\alpha'(t)\right) = 2\alpha'(t)\cdot\alpha''(t) = 0$$

since $\alpha'' \perp$ all tangent vectors, including α'.

(b) Give the definition of geodesic curvature for a general curve on a surface.

$$v^2 \cdot \kappa_g = \alpha''(t)\cdot J(T), \quad \text{where}$$

$v = |\alpha'(t)|$ and $J(T) = 90°$ rotation of $T = U \times T$.

(c) Prove that a geodesic has zero geodesic curvature.

$$\kappa_g = \frac{1}{v^2}\alpha'' \cdot J(T) = 0 \text{ since}$$

$\alpha'' \perp$ all tangent vectors, including $J(T)$.

(d) Suppose α is a geodesic on the standard unit sphere. Prove that it is contained in a plane.

1. The great circles (like the equator) are all geodesics by symmetry since they are the intersection of the unit sphere and planes thru the origin.
2. All initial positions and initial directions are achieved by great circles.
3. By uniqueness, all geodesics are thus great circles, which are contained in planes.

Problem 6. Suppose α is a geodesic on M and is also contained in a plane P. Prove that α is also a line of curvature of M. (Recall that a line of curvature is any curve whose tangent direction T is an eigenvector of the shape operator at every point.)

Without loss of generality, suppose $\alpha(s)$ is a unit speed geodesic. Then

$$\alpha''(s) = T'(s) = \kappa N \perp M.$$

Hence, we may choose $U = N$.

Since α is in a plane, $\tau = 0$. Hence

$$N'(s) = -\kappa T + \tau B$$

$$U'(s) = -\kappa T$$

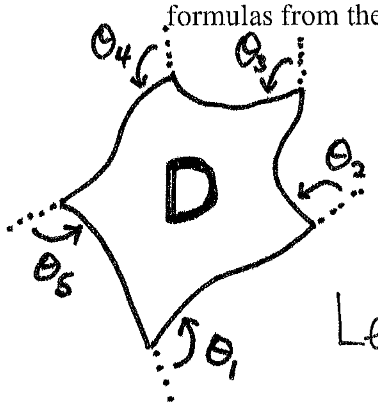
$$\nabla_T U = -\kappa T$$

$$S_p(T) = \kappa T$$

Hence, T is an eigenvector of the shape operator, so α is a line of curvature of M.

Problem 7. Let D be a "polygonal" subset of the flat xy plane, where the boundary of D is a simple closed curve which is smooth, except at a finite number of corners. That is, D is a polygon, but where the sides need not be straight, but instead are smooth curves that do not intersect.

State and prove the "Gauss-Bonnet theorem with corners" for D, directly without using any formulas from the book, using the fact that the Gauss curvature of D is zero.



We need to prove that

$$\left(\int_{\partial D} \kappa_g \, ds\right) + \sum_j \theta_j = 2\pi.$$

Let ∂D be parametrized by $\beta(s)$ with unit speed.

Define $\theta(s)$ such that $\beta'(s) = (\cos\theta(s), \sin\theta(s))$. Then

$J(T) = 90°$ rotation of T

$$\boxed{\kappa_g} = \beta''(s) \cdot (U \times T)$$

$$= (-\sin\theta(s), \cos\theta(s))\,\theta'(s) \cdot (-\sin\theta(s), \cos\theta(s))$$

$$= \boxed{\theta'(s)}$$

Hence, since $\theta(s)$ ~~changes~~ increases by 2π going around once,

$$2\pi = \int_{\partial D} \theta'(s)\,ds + \sum_j \theta_j$$

$$= \int_{\partial D} \kappa_g \, ds + \sum_j \theta_j.$$

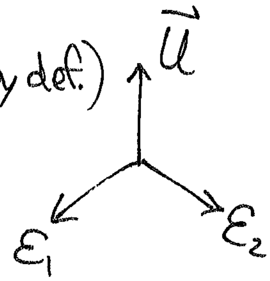
Problem 8. Given a coordinate chart with $F = 0$, define $\vec{\mathcal{E}}_1 = \vec{x}_u/\sqrt{E}$ and $\vec{\mathcal{E}}_2 = \vec{x}_v/\sqrt{G}$ as an orthonormal frame on the surface, as usual. Let $\beta(s)$ be any unit speed curve. Using the definition of ω_{21} from the book (by John Oprea), prove that

$$\kappa_g = \frac{d\theta}{ds} + \omega_{21}(\beta'(s)),$$

where $\theta(s)$ is the angle that $\beta'(s)$ makes with $\vec{\mathcal{E}}_1$ (in the direction of $\vec{\mathcal{E}}_2$).

$\vec{T} = \beta'(s) = \cos\theta(s) \cdot \vec{\mathcal{E}}_1 + \sin\theta(s) \cdot \vec{\mathcal{E}}_2$ (by def.)

$\vec{U} = \vec{\mathcal{E}}_1 \times \vec{\mathcal{E}}_2$



$\vec{U} \times \vec{T} = \vec{U} \times \left(\cos\theta(s)\, \vec{\mathcal{E}}_1 + \sin\theta(s)\, \vec{\mathcal{E}}_2 \right)$

$= \cos\theta(s) \cdot \vec{\mathcal{E}}_2 - \sin\theta(s)\, \vec{\mathcal{E}}_1$

$$\boxed{\begin{aligned} \nabla_{\beta'(s)} \vec{\mathcal{E}}_1 &= \omega_{21}(\beta'(s))\, \vec{\mathcal{E}}_2 \\ \nabla_{\beta'(s)} \vec{\mathcal{E}}_2 &= -\omega_{21}(\beta'(s))\, \vec{\mathcal{E}}_1 \end{aligned}}$$ by definition

After simplifying and plugging in, we get

$$\beta''(s) = \left(\theta'(s) + \omega_{21}(\beta'(s))\right)\left(-\sin\theta(s)\, \vec{\mathcal{E}}_1 + \cos\theta(s)\, \vec{\mathcal{E}}_2\right)$$

$$= \kappa_g \cdot (U \times T)$$ (by def. of κ_g)

Thus,

$$\kappa_g = \theta'(s) + \omega_{21}(\beta'(s))$$

Problem 9. Prove the Gauss Bonnet Theorem for a Disk (without corners) using the identity from problem 8. The only theorem from the book you may quote without proof is the formula for the Gauss curvature,

$$K = -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right].$$

for $F=0$, which we will assume.

We will use $0 = F_u = (X_u \cdot X_v)_u = X_{uu} \cdot X_v + X_u \cdot X_{uv} \Rightarrow$

$\boxed{X_{uu} \cdot X_v} = -X_u \cdot X_{uv} = -\frac{1}{2}(X_u \cdot X_u)_v = \boxed{-\frac{1}{2}E_v}$ and

$\boxed{X_{uv} \cdot X_v} = \frac{1}{2}(X_v \cdot X_v)_u = \boxed{\frac{1}{2}G_u}$.

$$\beta(s) = \vec{X}(u(s), v(s))$$

$$\beta'(s) = u'(s)\vec{X}_u + v'(s)\vec{X}_v$$

$$\omega_{21}(\beta'(s)) = \left(\nabla_{\beta'(s)}\vec{\mathcal{E}}_1\right) \cdot \vec{e}_2 = \left[u'(s)\left(\frac{\vec{X}_u}{\sqrt{E}}\right)_u + v'(s)\left(\frac{\vec{X}_u}{\sqrt{E}}\right)_v\right] \cdot \frac{\vec{X}_v}{\sqrt{G}}$$

$$= u'(s)\frac{\vec{X}_{uu} \cdot \vec{X}_v}{\sqrt{EG}} + v'(s)\frac{\vec{X}_{uv} \cdot \vec{X}_v}{\sqrt{EG}} = u'(s)\left(-\frac{E_v}{2\sqrt{EG}}\right) + v'(s)\left(\frac{G_u}{2\sqrt{EG}}\right)$$

$$2\pi - \int_{\partial D} \kappa_g\, ds = \int_{\partial D}\left(\frac{d\theta}{ds} - \kappa_g\right)ds = \int_{\partial D} -\omega_{21}(\beta'(s))\,ds$$

$$= \int_{\partial D} \frac{E_v}{2\sqrt{EG}}\,du - \frac{G_u}{2\sqrt{EG}}\,dv \underset{\text{Thm}}{\overset{\text{Green's}}{=}} \iint_D \left[-\left(\frac{E_v}{2\sqrt{EG}}\right)_v - \left(\frac{G_u}{2\sqrt{EG}}\right)_u\right] du\,dv$$

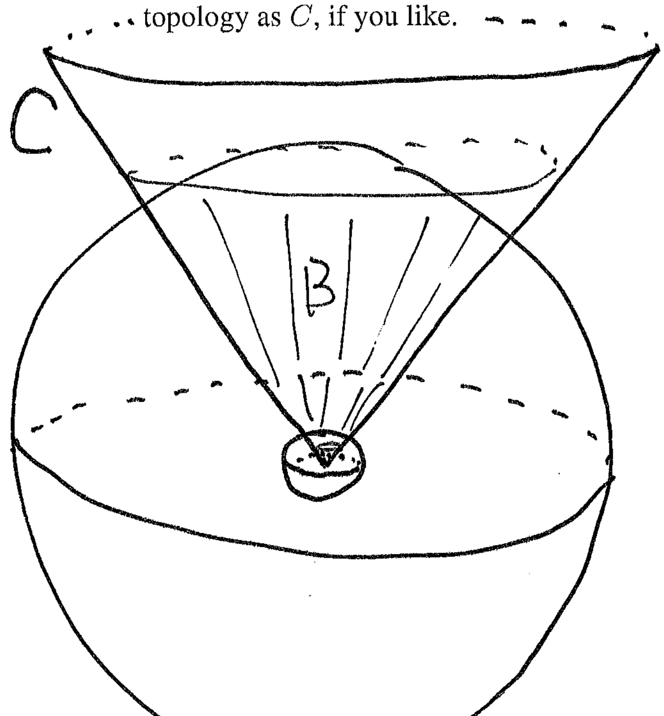
$$= \iint_D K \cdot \left(\sqrt{EG}\,du\,dv\right)$$

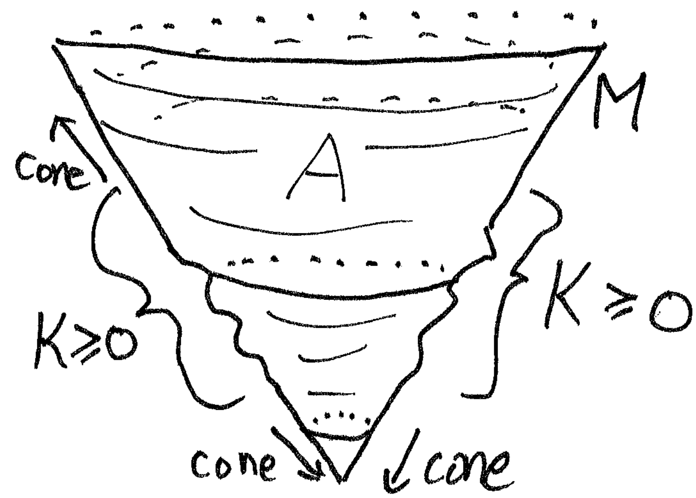
$$= \iint_D K\,dA. \Longrightarrow \iint_D K\,dA + \int_{\partial D} \kappa_g\,ds = 2\pi$$

Problem 10. (The Rigidity of the Cone)

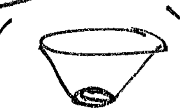
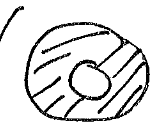
Let C be a circular cone in R^3 whose vertex is at the origin. Let M be the same as C inside the sphere of radius 1 centered at the origin and outside the sphere of radius 100 centered at the origin. Suppose M is a smooth surface everywhere as well (except at the vertex at the origin).

Prove that if M has Gauss curvature $K \geq 0$ everywhere, than M must have $K = 0$ everywhere (except at the origin where C and M are not smooth). You may assume that M has the same topology as C, if you like.





$$A = M \cap \left(B_{101}(0) \setminus B_{1/2}(0)\right)$$
$$B = C \cap \left(B_{101}(0) \setminus B_{1/2}(0)\right)$$

$\chi(A) = \chi($  $) = \chi($  $) = 1 - 1 = 0.$

By Gauss-Bonnet,

$$\iint_A K\,dA + \int_{\partial A} \kappa_g\,ds = 2\pi\chi(A) = 0$$

is also true for B, where $K_B = 0 \Rightarrow \int_{\partial B} \kappa_g\,ds = 0$

$\Rightarrow \int_{\partial A} \kappa_g\,ds = 0 \Rightarrow$

$\boxed{\iint_A K\,dA = 0}$. But $K \geq 0$, so $\boxed{K \equiv 0}$.

Problem 11. For this problem, you may use the first variation of area formula

$$A'(0) = -\int_M 2H\,(\vec{U}\cdot\vec{V})\,dA,$$

where $A(s)$ is the area of the surface $M(s)$ which, at $s = 0$, has unit normal vector $\vec{U}$ and mean curvature H, and is flowing with velocity $\vec{V}$ as s increases. (The above formula is true as long as $\vec{V} = 0$ on the boundary of M, which is all you will need for this problem.)

(a) Suppose that M is a smooth, area minimizing surface (meaning that any other surface with the same boundary has at least as much area). Prove that M has constant mean curvature.

Minimizer for area $\Rightarrow$ critical point for area $\Rightarrow$

$0 = A'(0) = \int_M -2H(\vec{u}\cdot\vec{V})\,dA$ for <u>all</u> V $\Rightarrow$

$\boxed{H=0}$ everywhere

since $\vec{U}\cdot\vec{V}$ could be anything. For example, let $\vec{V} = H\vec{u}$ $\Rightarrow$

$0 = \int_M -2H^2\,dA \Rightarrow \boxed{H=0.}$

(b) If the above surface M has area 5, what is the value of its mean curvature everywhere?

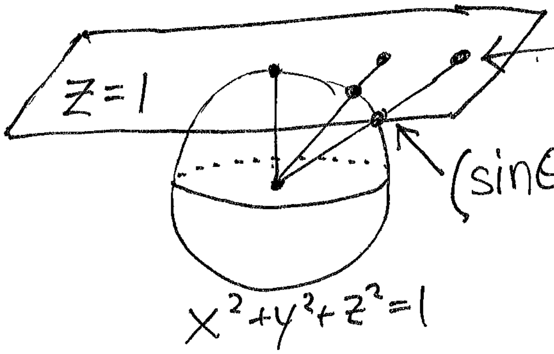
$\underbrace{\text{If the above surface } M \text{ has area 5}}_{\text{irrelevant.}}$

We've already proved that $H=0$.

Problem 12. It turns out that it is possible to define coordinates (u, v) on some portion of the Earth so that geodesics on the surface of the Earth appear as straight lines in the (u, v) coordinate chart.

(a) Define u and v in terms of the usual lattitude and longitude coordinates (θ, ϕ). Your coordinates do not need to cover the whole Earth. There is more than one correct answer, but be sure to prove that your answer is correct.

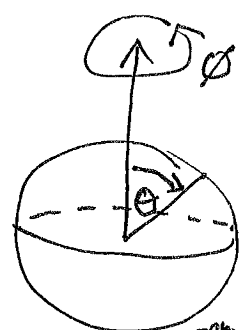
geodesics on the Earth = great circles = (planes thru origin) $\cap S^2$. Hence projecting to a plane through origin will map great circles to lines in the plane:



$(\tan\theta\cos\phi, \tan\theta\sin\phi, 1) = (u, v, 1)$

↑ (project by scaling)

$(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$



$$\boxed{\begin{aligned} u &= \tan\theta\cos\phi \\ v &= \tan\theta\sin\phi \end{aligned}}$$

(Technically, lattitude is $\pi/2 - \theta$.)

(b) What fraction of the surface area of the Earth does your coordinate chart cover?

$\frac{1}{2}$ the Earth (the northern hemisphere in this case.)